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Periodic Review Inventory Model with Controllable Lead Time under Service level Constraint where Backorder Rate depends on Protection Interval

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Abstract

In this paper, a periodic review inventory model with controllable lead time under the service level constraint has been examined where backorder rate is dependent on the back order discount and the length of the protection interval which is sum of the review period and lead time. Shortages are allowed and partially backlogged. The demand during the protection interval is assumed to be normally distributed. The total cost has been obtained by jointly optimizing the review period, lead time and backorder discount under the effect of known service level. This study not only provides the greater flexibility to the inventory manager to coordinate inventory in more efficient manner but also makes enormous savings in total expected cost.

Keywords: Inventory, Periodic review, Crashing cost, Lead-Time, Backorder discount, Service level

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1. Introduction

In Inventory management, lead time plays a very important role. In most of the literature dealing with inventory problems, in either deterministic or the probabilistic models, the lead time is viewed as a prescribed constant or a stochastic variable, which therefore is not subject to control (Silver and Peterson (1985)). Lead time usually consists of: order preparation, order transit, supplier lead time, delivery time, and setup time (Tersine (1982)). In many practical situations, the lead time can be reduced at an added crashing cost; in other words, it is controllable. By shortening the lead time, we can lower the safety stock, reduce the stock out loss and improve the customer service level so as to increase the competitive edge in business.

For Consumer: quality, price, delivery time and service are the most significant elements for any product. Out of these, delivery time has a special role to play. Now a day, each firm tries to provide fastest delivery of items and helps to adjust with high fluctuation of demand and shorter product life cycle. That is how this particular element grabs our concentration and in the meantime is widely recognized in most real-life applications as a decision variable. For fast delivery, suppliers work overtime, add manpower, and advance equipment, reset layout or choose better logistic means to cut down the lead-time in order to supply their products to their valued customer as quickly as possible to get the edge over their competitions.

Ben-Daya and Raouf (1994) presented a continuous review model in which they considered both the lead time and the order quantity as decision variables where the shortages are neglected. They provided optimal lead time and optimal order quantity which minimizes the sum of the ordering cost, inventory holding cost and lead time crashing cost. In a recent paper, Wu and Tsai (2001) have extended the Ben-Daya and Raouf (1994) model by adding the stock out cost. In their studies (2001), they assumed a fixed stock-out cost attributable to each stock-out occasion.

In addition, it is also observed that many products of famous brands or fashionable goods such as leather shoes, hi-fi equipment, cosmetics and clothes may lead to a situation in which customers prefer their demands to be backordered when shortages occur (see Lin, Y. J. (2008), Montgomery, D. C. et. al. (1973), Pan C. H. and Y. C. Hsiao (2001)). In this case, a greater degree of efficiency in supply is to be maintained by the suppliers because any inordinate delay in supply may compel the customers to cancel their back orders. This phenomenon reveals that, as shortages occur, longer the length of lead time is, larger the amount of shortages is, smaller the proportion of customers can wait and hence smaller the backorder rate would be. Under this situation, for a supplier, how to control an appropriate length of lead time to determine a target value of backorder rate so as to minimize the inventory relevant cost and increase the competitive edge in business is worth discussing. Consequently, we here assume that the backorder rate is dependent on the length of lead time through the amount of shortages and let the backorder rate be a control variable. Moreover, we also consider the objective function with a service level constraint which implies that the stock-out level per cycle is bounded (see Aardal K. et. al. (1989), Chuang, B. R. et. al. (2004), Chu P. et. Al. (2005), Lee W. C. et. al. (2006), Moon, I and Choi, S. (1994), Ouyang L. Y. and Wu K. S. (1997), Ouyang L. Y. and Chuang B. R. (2000), Liang S. K., et. al. (2008)).

In this study, lead time, review period and backorder discount are considered as decision variables with a mixture of backorders and lost sales inventory model where backorder rate depends on protection interval. Instead of having a stock-out cost term in the objective function, a service level constraint, which implies that the stock–out level per cycle is bounded, is added to the model. We first assume that the protection interval (i.e. the review period plus the lead time) demand follows a normal distribution and find the optimal ordering policy. Then, we consider that the protection interval demand has only known finite first and second moments (and hence, the mean and variance are also known and finite) but make no assumption on its probability distribution. We solve this inventory model by using the minimax distribution free approach. Furthermore, a numerical example is also given to illustrate the results.

2. Notation and Assumption

To develop the proposed model, we adopt the following notation and assumptions in this paper.

2.1 Notation

D : Average demand per year

K : Fixed ordering cost per inventory cycle

h : Inventory holding cost per unit per year

R : Target Level

β : Fraction of the demand back ordered during stock out period such as $0 \le \beta \le 1$

β_0 : Upper bound of the backorder ratio

π_0 : Marginal Profit (i.e. cost of lost demand) per unit

π_x : Back order price discount offered by the supplier per unit

L : Length of lead-Time

X : Protection interval demand which has a p. d. f. f_X with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$ (>0) for the protection interval $(T+L)$ where σ denotes standard deviation of the demand per unit.

Ω : The class of p. d. f. f_X of the protection interval demand with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$

A : Safety factor

T : Length of a review period

α : The proportion of demand that are not met from stock, which means $(1-\alpha)$ represents service level where $0<\alpha<1$

$E(.)$: Mathematical Expectation

X^+ : Maximum value of x and 0 i.e., $X^+ = \max\{x, 0\}$

EAC : Expected annual cost

EAC^W : Least upper bound of expected annual cost.

2.2 Assumptions

1. The inventory level is reviewed every T units of time. A sufficient quantity is ordered up to the target level R, and the ordering quantity is arrived after L units of time.
2. The length of the lead-time L does not exceed an inventory cycle time T so that there is never more than a single order outstanding in any cycle.
3. The target level R = Expected demand during the protection interval + safety stock (SS) i.e. $R = D(T+L) + A\sigma\sqrt{(T+L)}$ where A is the safety factor and satisfies $P[x>R]=q$, q represent the allowable stock out probability which means service level is defined during the protection interval and is given.
4. The lead-time L consists of n mutually independent components. The i[th] component has a minimum duration a_i and normal duration b_i, and a crashing

cost per unit time c_i. Arranging c_i such that $c_1 \le c_2 \le c_3 \le ... \le c_n$ for the convenience. Since it is clear that the reduction of lead-time should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.

5. Let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead time with components 1,2,…,i crashed to their minimum duration, then L_i can be expressed as $L_i = L_0 - \sum_{j=1}^{i}(b_j - a_j)$, , i= 1,2,…,n and the lead time crashing cost per cycle $C(L)$ is given as $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.

6. Assuming that a fraction β $(0 \le \beta \le 1)$ of the demand during the stock out period can be backordered so the remaining fraction $(1-\beta)$ is lost. The backorder rate β is variable and is in proportion to the price discount π_x offered by the supplier per unit and the protection interval.

   Thus $\beta = \dfrac{T\beta_0\pi_x}{(T+L)\pi_0}$ where $0 \le \beta_0 \le 1$ and $0 \le \pi_x \le \pi_0$, therefore, our model is different from the previous models.

7. The service level constraint is $\dfrac{E(X-R)^+}{D(T+L)} \le \alpha$

3. Mathematical Model

We have assumed that the protection interval demand X has a p. d. f. f_X with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{(T+L)}$ with the target level $R = D(T+L) + A\sigma\sqrt{(T+L)}$ where A is already defined. As Montgomery (1973) proposed the periodic review model where the expected net inventory at the beginning of the period is $R - DL + (1-\beta)E(X-R)^+$. Therefore, the expected net inventory at the end of the period is $R - DL - DT + (1-\beta)E(X-R)^+$ which gives the expected holding cost per year as $h\left[R - DL - \dfrac{DT}{2} + (1-\beta)E(X-R)^+\right]$.

Now, the expected stock out cost per year is $\dfrac{\pi_x \beta E(X-R)^{+} + \pi_0(1-\beta)E(X-R)^{+}}{T}$ where

$E(X-R)^{+}$ is the expected demand shortage at the end of cycle i.e., $E(X-R)^{+}$ $= \int_R^{\infty} (x-R) f_X dx$.

When the lead time L is reduced to L_i then the Annual lead time crashing cost $= \dfrac{C(L_i)}{T}$

Now the objective is to minimize the total expected annual cost (EAC) which is the sum of

= Ordering cost + stock out cost + holding cost + lead-time crashing cost

Subject to the constraint on expected shortages.

$$Min\left[EAC(T,\pi_x,L\right] = \frac{K}{T} + \frac{\pi_x \beta E(X-R)^{+} + \pi_0(1-\beta)E(X-R)^{+}}{T}$$

$$+h\left[R - DL - \frac{DT}{2} + (1-\beta)E(X-R)^{+}\right]$$

$$+\frac{C(L_i)}{T}$$

$$subject\ to \frac{E(X-R)^{+}}{D(T+L)} \le \alpha \qquad \ldots(1)$$

Also, we have assumed that the backorder rate β depends on the backorder price discount π_x and protection interval $(T+L)$. Thus $\beta = \dfrac{T\beta_0 \pi_x}{(T+L)\pi_0}$ and $R = D(T+L) + A\sigma\sqrt{(T+L)}$, where A is safety factor. The equation (1) can be written as

$$Min\left[EAC(T,\pi_x,L\right] = \frac{K + C(L_i)}{T} + \frac{E(X-R)^{+}\left[\dfrac{T\beta_0 \pi_x^2}{(T+L)\pi_0} + \pi_0\left(1 - \dfrac{T\beta_0 \pi_x}{(T+L)\pi_0}\right)\right]}{T}$$

$$+h\left[\frac{DT}{2}+A\sigma\sqrt{T+L}+\left(1-\frac{T\beta_0\pi_x}{(T+L)\pi_0}\right)E(X-R)^+\right]$$

i.e.

a. Normal distribution

b. Unknown distribution

3.1 Lead Time Demand with Normal Distribution

In this section, we have assumed that the probability distribution of protection interval demand X has a normal distribution with mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$

So, the expected shortages occurring at the end of the cycle is given by

$$E(X-R)^+ = \int_R^\infty (x-R)f_X dx = \sigma\sqrt{T+L}\ \psi(A) > 0$$

Where $\psi(A)=\phi(A)-A[1-\Phi(A)]$, ϕ and Φ are the standard normal p. d. f. and c. d. f., respectively.

Therefore, equation (2) is reduced to

$$Min\left[EAC(T,\pi_x,L\right] = \frac{K+C(L_i)}{T}$$

$$+\frac{\sigma\sqrt{T+L}\ \psi(A)\left[\frac{T\beta_0\pi_x^2}{(T+L)\pi_0}+\pi_0\left(1-\frac{T\beta_0\pi_x}{(T+L)\pi_0}\right)\right]}{T}$$

$$+h\left[\frac{DT}{2}+A\sigma\sqrt{T+L}+\left(1-\frac{T\beta_0\pi_x}{(T+L)\pi_0}\right)\sigma\ \sqrt{T+L}\ \Psi(A)\right]$$

$$subject\ to \frac{\sigma\psi(A)}{D\sqrt{T+L}} \le \alpha \qquad \ldots(3)$$

The model (3) is a nonlinear programming problem; it can be verified that the Kuhn-tucker conditions are not satisfied. In order to solve this kind of problem, we first ignore the service level constraint and for fixed T and π_x take the first and second partial derivatives of $EAC(T,\pi_x,L)$ with respect to L.

$$\frac{\partial EAC(T,\pi_x,L)}{\partial L}=$$

$$\frac{C_i}{T}+\left[\frac{hA\sigma}{2}(T+L)^{-\frac{1}{2}}+\left(\frac{1}{2}(T+L)^{-\frac{1}{2}}+\frac{T\beta_0\pi_x}{2\pi_0}\left(T+L\right)^{-\frac{3}{2}}\right)h\sigma\,\psi(A)\right]$$

$$+\sigma\psi(A)\left[\frac{-\beta_0\pi_x^2}{2\pi_0}(T+L)^{-\frac{3}{2}}+\frac{\pi_0}{2T}(T+L)^{-\frac{1}{2}}+\frac{\beta_0\pi_x}{2}\left(T+L\right)^{-\frac{3}{2}}\right]$$

$$\frac{\partial^2 EAC(T,\pi_x,L)}{\partial L^2}=.$$

$$-h\left[\frac{1}{4}A\sigma\left(T+L\right)^{-\frac{3}{2}}+\left(\frac{1}{4}(T+L)^{-\frac{3}{2}}+\frac{3}{4}\frac{T\beta_0\pi_x}{\pi_0}\left(T+L\right)^{-\frac{5}{2}}\right)\sigma\,\psi(A)\right]$$

$$-\sigma\,\psi(A)\left[\pi_0\left(\frac{1}{4T}(T+L)^{-\frac{3}{2}}+\frac{3}{4}\frac{\beta_0\pi_x}{\pi_0}(T+L)^{-\frac{5}{2}}\right)-\frac{3}{4}\frac{\beta_0\pi_x^2}{\pi_0}(T+L)^{-\frac{5}{2}}\right]<0$$

Hence, for fixed (T,π_x), $EAC(T,\pi_x,L)$ is a concave function of $L\in\left(L_i,L_{i-1}\right)$ because $\frac{\partial^2 EAC(T,\pi_x,L)}{\partial L^2}<0$. Therefore, for fixed (T,π_x), the minimum total expected annual cost will occur at the end points of the interval $\left(L_i,L_{i-1}\right)$ and the optimal solution must obey the service level constraint. On the other hand, it can be shown that $EAC(T,\pi_x,L)$ is convex in (T,π_x) [Appendix 1].

Thus for fixed $L\in\left(L_i,L_{i-1}\right)$, the minimum value of $EAC\,(T,\pi_x,L)$ will occur at the point (T,π_x) that satisfy $\frac{\partial EAC(T,\pi_x,L)}{\partial T}=0$ and $\frac{\partial EAC(T,\pi_x,L)}{\partial \pi_x}=0$.

Now, $\frac{\partial EAC(T,\pi_x,L)}{\partial T}=0\Rightarrow-\frac{K+C(L_i)}{T^2}$

$$+\sigma\psi(A)\left[-\frac{1}{2}\frac{\beta_0\pi_x^2(T+L)^{-\frac{3}{2}}}{\pi_0}+\pi_0\left\{-\frac{(T+L)^{\frac{1}{2}}}{T^2}+\frac{(T+L)^{-\frac{1}{2}}}{2T}\right\}+\frac{\beta_0\pi_x(T+L))^{-\frac{3}{2}}}{2}\right]$$

$$+h\left[\frac{D}{2}+\frac{A\sigma}{2}(T+L)^{-\frac{1}{2}}+\left(\frac{1}{2}(T+L)^{-\frac{1}{2}}-\frac{\beta_0\pi_x(T+2L)(T+L)^{-\frac{3}{2}}}{2\pi_0}\right)\sigma\psi(A)\right]$$

$$\ldots(4)$$

This can be written as

$$\frac{K+C(L_i)}{T^2}$$

$$+\sigma\psi(A)\left[\frac{1}{2}\frac{\beta_0\pi_x^2(T+L)^{-\frac{3}{2}}}{\pi_0}+\pi_0\left\{\frac{(T+L)^{\frac{1}{2}}}{T^2}-\frac{(T+L)^{-\frac{1}{2}}}{2T}\right\}-\frac{\beta_0\pi_x(T+L))^{-\frac{3}{2}}}{2}\right]$$

$$=h\left[\frac{D}{2}+\frac{A\sigma}{2}(T+L)^{-\frac{1}{2}}+\left(\frac{1}{2}(T+L)^{-\frac{1}{2}}-\frac{\beta_0\pi_x(T+2L)(T+L)^{-\frac{3}{2}}}{2\pi_0}\right)\sigma\psi(A)\right]$$

$$\ldots(5)$$

Where $\frac{\partial EAC(T,\pi_x,L)}{\partial \pi_x}=0 \Rightarrow \pi_x=\frac{\pi_0+Th}{2}$ ...(6)

Since it is difficult to obtain the solution for T and π_x explicitly as the evolution of equation (5) and equation (6) need the value of each other. As a result, we must establish the following iterative algorithm to find the optimal (T,π_x).

Algorithm 1

Step 1. For a given allowable stock-out probability, we can find the value of safety factor A by checking the standard normal table, and thus obtain the value of $\psi(A)$.

Step 2. For given A, $\psi(A)$ and each L_i, $i = 0,1,2,...,n,$ utilize a numerical search technique to compute T_i and π_x from (5) and (6).

Step 3. For each pair (T_i, π_{x_i}, L_i), compute the corresponding total expected annual cost $EAC(T_i, \pi_{xi}, L_i), i = 0,1,2,...,n$ where $E(X-R)^+ = \sigma\sqrt{(T_S + L_S)}\psi(A)$ satisfies the service level constraints of equation (3) that is check $\sigma\,\psi(A)/D\sqrt{(T_S + L_S)} \le \alpha.$

Step 4. Find $EAC(T_S, \pi_{x_S}, L_S) = \min_{i=0,1,2,...,n}\left[EAC(T_i, \pi_{x_i}, L_i)\right]$ where service level constraint is satisfied.

Then, $EAC(T_S, \pi_{x_S}, L_S)$ is the local optimum solution.

3.2 Lead Time Demand with Unknown Distribution

If the lead time demand does not follow normal distribution or the probability distribution is unknown with first two moments, then the solution can be obtained by minimax approach. Since the probability distribution of X is unknown, we cannot find the exact value of $E(X-R)^+$. Now we use a minimax distribution free procedure to solve

$$\min_{T>0,\pi_x>0,L>0}\ \max_{F\in\Omega}\ EAC(T,\pi_x,L)$$

$$subject\ to\ \frac{E(X-R)^+}{D(T+L)} \le \alpha$$

We need the following proposition to shorten the problem which was asserted by Gallego and Moon (1993).

Proposition 1 For any $F \in \Omega$,

$$E(X-R)^+ \le \frac{1}{2}\{\sqrt{\sigma^2(T+L) + (R - D(T+L))^2} - [R - D(T+L)]\}$$

$$\le \frac{1}{2}\,\sigma\sqrt{(T+L)}\ \ (\sqrt{1+A^2} - A)$$

Moreover, the upper bound (8) is tight.

Then the equation (7) can be reduced to

$$Min\left[EAC^{W}(T,\pi_x,L\right]=\frac{K+C(L_i)}{T}$$

$$+\frac{\sigma\sqrt{T+L}\ (\sqrt{1+A^2}-A)\left[\frac{T\beta_0\pi_x^2}{(T+L)\pi_0}+\pi_0\left(1-\frac{T\beta_0\pi_x}{(T+L)\pi_0}\right)\right]}{2T}$$

$$+\frac{h}{2}\left[\frac{DT}{2}+A\sigma\sqrt{T+L}+\left(1-\frac{T\beta_0\pi_x}{(T+L)\pi_0}\right)\sigma\ \sqrt{T+L}\ \ (\sqrt{1+A^2}-A)\right]$$

$$subject\ to\frac{\sigma(\sqrt{1+A^2}-A)}{2D\sqrt{T+L}}\le\alpha \qquad \ldots(9)$$

Where $EAC^{W}(T,\pi_x,L)$ is the least upper bound of $EAC(T,\pi_x,L)$.

As notified in the preceding section, we first ignore the service level constraint. Therefore, the first order conditions are necessary and sufficient conditions for optimality. Using the first condition of derivatives, we get

$$\frac{\partial EAC^{W}(T,\pi_x,L)}{\partial T}=0\Rightarrow-\frac{K+C(L_i)}{T^2}$$

$$+\frac{\sigma(\sqrt{1+A^2}-A)}{2}\left[-\frac{1}{2}\frac{\beta_0\pi_x^2(T+L)^{-\frac{3}{2}}}{\pi_0}+\pi_0\left\{-\frac{(T+L)^{\frac{1}{2}}}{T^2}+\frac{(T+L)^{-\frac{1}{2}}}{2T}\right\}+\frac{\beta_0\pi_x(T+L))^{-\frac{3}{2}}}{2}\right]$$

$$+h\left[\frac{D}{2}+\frac{A\sigma}{2}(T+L)^{-\frac{1}{2}}+\left(\frac{1}{2}(T+L)^{-\frac{1}{2}}-\frac{\beta_0\pi_x(T+2L)(T+L)^{-\frac{3}{2}}}{2\pi_0}\right)\frac{\sigma(\sqrt{1+A^2}-A)}{2}\right] \qquad \ldots(10)$$

and

$$\pi_x=\frac{\pi_0+hT}{2}$$

Since it is difficult to obtain the exact value of service factor A which depends upon the required service level on the basis of allowable stock out probability q, because the p. d. f. $f_X(x)$ is unknown. So, the following proposition has been used to find accurate value of A.

Therefore, the algorithm to find the optimal review period, lead-time and backorder discount

can be established by using the proposition given below:

Proposition 2

Let X represent the protection interval demand that has p. d. f. $f_X(x)$ with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$ then for any real number c > 0,

$$p[X > c] \le \frac{\sigma^2(T+L)}{\sigma^2(T+L)+[c-D(T+L)]^2}.$$

If we take R instead of c, then $p[X > R] \le \dfrac{\sigma^2(T+L)}{\sigma^2(T+L)+[R-D(T+L)]^2}$ which implies

$$p[X > R] \le \frac{1}{1+A^2}, \quad q = p[X > R] \le \frac{1}{1+A^2}$$

Hence, $q = \dfrac{1}{1+A^2}$, Therefore $A \in \left[0, \sqrt{\dfrac{1}{q}-1}\right]$

Algorithm 2

Step 1. For given q, we divide the interval $\left[0, \sqrt{\dfrac{1}{q}-1}\right]$ into N equal subintervals, where N is large enough. Let $A_0 = 0$, $A_N = \sqrt{\dfrac{1}{q}-1}$ and

$$A_j = A_{j=1} + \frac{A_N - A_0}{N}, \ j = 1, 2, ..., \ N-1$$

Step 2. For each L_i $(i = 0, 1, 2, ..., n)$ and for given $A_j \in \{A_0, A_1, ..., A_N\}$, $j = 0, 1, 2, ..., N$, using numerical search technique, to evaluate T_{i,A_j} and π_{x_i, A_j} from equation (10) and (11).

Step 3. For each L_i $(i = 0,1,2,...,n)$, Compute the corresponding expected annual cost EAC^W $(T_{i,A_j}, \pi_{x_i,A_j}, L_i)$, $j = 0,1,2,....m$

Step 4. Find $\min_{A_j \in \{A_0, A_1, ..., A_N\}} EAC^W (T_{i,A_j}, \pi_{x_i,A_j}, L_i)$,

If $EAC^W (T_{i,A_{s_i}}, \pi_{x_i,A_{s_i}}, L_{i,}) = \min_{A_l \in \{A_0, A_1, ..., A_N\}} EAC^W (T_{i,A_j}, \pi_{x_i,A_j}, L_i)$, and if the values of A_{s_i} and $(T_{i,A_{s_i}}, \pi_{x_i,A_{s_i}}, L_{i,})$ satisfy the service level constraint of model (9) then $(T_{i,A_{s_i}}, \pi_{x_i,A_{s_i}}, L_{i,})$ is the local optimal solution and A_{s_i} is the safety factor.

Step 5. Find $\min_{i=0,12,...,n} EAC^W (T_{i,A_{s_i}}, \pi_{x_i,A_{s_i}}, L_{i,})$,

If $EAC^W (T_{u\,A_{su}}, \pi_{x_u,A_{s(u)}}, L_{u,}) = \min_{i=0,12,...,n} EAC^W (T_{i,A_{s_i}}, \pi_{x_i,A_{s_i}}, L_{i,})$

Then $EAC^W (T_{u\,A_{su}}, \pi_{x_u,A_{s(u)}}, L_{u,})$ is the optimal solution and $A_{s(u)}$ is the optimal safety factor.

4. Numerical Example

In order to illustrate the solution algorithms, we have considered an inventory system with having data D=600 units per year, K= \$ 300 per order, σ = 7 units per week, π_0 = \$ 100 per unit, h= \$ 20 per unit per year, q = 0.2 where A_0= 0 and A_N = 2, N=200.

The lead-time has three components, which have been shown in table 1. We have solved the cases for different upper bounds of the backorder ratio β = 0, 0.1, 0.2, 0.3. Then applying the algorithm 1, crashing has been carried out for lead-time for different backorder ratio and illustrated in table 2.

Table 1: Lead time data

Lead time component i	Normal duration (days) b_i	Minimum duration (days) a_i	Unit Crashing cost per day, c_i
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 2: Optimal solution with service level of 98% ($\alpha = 0.02$) for Normal distribution

$\beta_0 = 0$						
i	L	T	π_x	R	β	EAC
1	8	12.01	52.31	295.06	0	3575.39
2	6	11.93	52.29	267.61	0	3483.83
3	4	11.83	52.28	239.80	0	**3421.16**
4	3	11.78	52.27	225.69	0	3495.37
$\beta_0 = 0.1$						
i	L	T	π_x	R	β	EAC
1	8	12.01	52.31	295.01	0	3531.73
2	6	11.93	52.29	267.58	0	3434.00
3	4	11.83	52.28	239.78	0	**3361.17**
4	3	11.78	52.27	225.68	0	3427.03
$\beta_0 = 0.2$						
i	L	T	π_x	R	β	EAC
1	8	11.85	52.28	301.76	0.00	3610.94
2	6	11.92	52.29	267.56	0.00	3384.19
3	4	11.83	52.28	239.77	0.01	**3301.19**
4	3	11.78	52.26	225.67	0.01	3358.71
$\beta_0 = 0.3$						
i	L	T	π_x	R	β	EAC
1	8	12.00	52.31	294.93	0.00	3444.47
2	6	12.00	52.31	268.56	0.01	3333.45
3	4	11.83	52.28	239.75	0.01	**3241.22**
4	3	11.78	52.26	225.66	0.01	3290.39
$\beta_0 = 0.4$						
i	L	T	π_x	R	β	EAC
1	8	12.00	52.31	294.89	0.01	3400.88
2	6	11.92	52.29	267.50	0.01	3284.62
3	4	11.83	52.27	239.74	0.01	**3181.26**
4	3	11.78	52.26	225.65	0.01	3222.08
$\beta_0 = 0.5$						
i	L	T	π_x	R	β	EAC
1	8	11.99	52.31	294.85	0.01	3357.31

2	6	11.92	52.29	267.47	0.01	3234.85
3	4	11.83	52.27	239.72	0.01	**3121.31**
4	3	11.78	52.26	225.64	0.02	3153.78
$\beta_0 = 1$						
i	L	T	π_x	R	β	EAC
1	8	11.98	52.30	294.64	0.01	3139.82
2	6	11.91	52.29	267.34	0.02	2986.32
3	4	11.82	52.27	239.65	0.03	2821.76
4	3	11.77	52.26	225.60	0.04	**2812.39**

As we know $\beta = 0$, represent complete lost sale case; $\beta = 1$, represent complete backorder case and $0 < \beta < 1$ represent partially backorder case. From table 2, it can be observed that the total expected cost is higher when $\beta_0 = 0$ (i.e. lost sale case) with the reduction of lead time as it involves extra investment for providing higher level of service. But, when $\beta_0 > 0$ then crashing of lead time is desirable as it helps to capture the backorder demand and reduces the intensity of shortages. Moreover, reduction of lead time is beneficial for certain level of backorder ratio else the cost of the system will increase as the profits generated by capturing the backorder demand would be less than the crashing cost of the lead time. The impact of reducing lead time on the system is very encouraging as it increases the level of backorder rate (β) also.

Table 3: Optimal solution with service level of 99% $(\alpha = 0.01)$ for Normal distribution

$\beta_0 = 0$						
i	L	T	π_x	R	β	EAC
1	8	11.86	52.28	301.80	0	3651.71
2	6	11.79	52.27	274.11	0	3550.41
3	4	11.72	52.25	246.07	0	**3476.62**
4	3	11.68	52.25	231.86	0	3545.11
$\beta_0 = 0.1$						
i	L	T	π_x	R	β	EAC
1	8	11.86	52.28	301.78	0.00	3631.32
2	6	11.79	52.27	274.10	0.00	3527.11
3	4	11.72	52.25	246.06	0.00	**3448.52**
4	3	11.68	52.25	231.86	0.00	3513.06
$\beta_0 = 0.2$						
i	L	T	π_x	R	β	EAC
1	8	11.85	52.28	301.76	0.00	3610.94
2	6	11.79	52.27	274.09	0.00	3503.81
3	4	11.72	52.25	246.06	0.01	**3420.42**
4	3	11.68	52.25	231.85	0.01	3481.02

$\beta_0 = 0.3$						
i	L	T	π_x	R	β	EAC
1	8	11.85	52.28	301.74	0.00	3590.56
2	6	11.79	52.27	274.07	0.01	3480.52
3	4	11.72	52.25	246.05	0.01	**3392.32**
4	3	11.68	52.25	231.85	0.01	3448.98
$\beta_0 = 0.4$						
i	L	T	π_x	R	β	EAC
1	8	11.85	52.28	301.72	0.01	3570.19
2	6	11.79	52.27	274.06	0.01	3457.23
3	4	11.79	52.27	247.00	0.01	**3363.10**
4	3	11.68	52.25	231.84	0.01	3416.94
$\beta_0 = 0.5$						
i	L	T	π_x	R	β	EAC
1	8	11.85	52.28	301.70	0.01	3549.83
2	6	11.79	52.27	274.05	0.01	3433.94
3	4	11.72	52.25	246.04	0.01	**3336.13**
4	3	11.68	52.25	231.84	0.02	3384.90
$\beta_0 = 1$						
i	L	T	π_x	R	β	EAC
1	8	11.84	52.28	301.61	0.01	3448.07
2	6	11.78	52.27	273.98	0.02	3317.57
3	4	11.72	52.25	246.00	0.03	**3195.71**
4	3	11.68	52.25	231.82	0.04	3224.74

Table 4: Optimal solution with service level of 96% $(\alpha = 0.04)$ for Normal distribution

$\beta_0 = 0$						
i	L	T	π_x	R	β	EAC
1	8	12.33	52.37	289.79	0.00	3533.05
2	6	12.21	52.35	262.37	0.00	3446.79
3	4	12.07	52.32	234.51	0.00	**3389.56**
4	3	11.99	52.30	220.33	0.00	3465.26
$\beta_0 = 0.1$						
i	L	T	π_x	R	β	EAC
1	8	12.32	52.37	289.70	0.00	3436.08
2	6	12.20	52.35	262.31	0.00	3336.46
3	4	12.07	52.32	234.48	0.00	**3257.25**
4	3	11.98	52.30	220.31	0.00	3314.91
$\beta_0 = 0.2$						
i	L	T	π_x	R	β	EAC
1	8	12.31	52.37	289.61	0.00	3339.23

2	6	12.20	52.35	262.25	0.00	3226.23
3	4	12.06	52.32	234.44	0.01	**3124.99**
4	3	11.98	52.30	220.29	0.01	3164.60
$\beta_0 = 0.3$						
i	L	T	π_x	R	β	EAC
1	8	12.31	52.37	289.52	0.00	3242.49
2	6	12.19	52.35	262.19	0.01	3116.08
3	4	12.06	52.32	234.41	0.01	**2992.80**
4	3	11.98	52.30	220.27	0.01	3014.33
$\beta_0 = 0.4$						
i	L	T	π_x	R	β	EAC
1	8	12.30	52.37	289.43	0.01	3145.87
2	6	12.19	52.34	262.13	0.01	3006.01
3	4	12.06	52.32	234.38	0.01	**2860.65**
4	3	11.98	52.30	220.25	0.01	2864.09
$\beta_0 = 0.5$						
i	L	T	π_x	R	β	EAC
1	8	12.29	52.36	289.34	0.01	3049.35
2	6	12.19	52.34	262.07	0.01	2896.02
3	4	12.06	52.32	234.35	0.01	2728.56
4	3	11.98	52.30	220.23	0.02	**2713.90**
$\beta_0 = 1$						
i	L	T	π_x	R	β	EAC
1	8	12.26	52.36	288.89	0.01	2568.49
2	6	12.16	52.34	261.78	0.02	2347.36
3	4	12.04	52.32	234.19	0.03	2068.94
4	3	11.97	52.30	220.12	0.04	**1963.54**

Table 5: Optimal solution with service level of 80% $(\alpha = 0.20)$ for Normal distribution

$\beta_0 = 0$						
i	L	T	π_x	R	β	EAC
1	8	15.42	52.97	298.71	0	3936.33
2	6	14.95	52.87	268.64	0	3805.94
3	4	14.37	52.76	237.21	0	**3687.00**
4	3	14.02	52.70	220.66	0	3710.06
$\beta_0 = 0.1$						
i	L	T	π_x	R	β	EAC
1	8	15.36	52.95	297.95	0.00	3084.05
2	6	14.91	52.87	268.15	0.00	2857.23
3	4	14.35	52.76	236.95	0.00	2581.95
4	3	14.01	52.69	220.49	0.00	**2478.22**

$\beta_0 = 0.2$						
i	L	T	π_x	R	β	EAC
1	8	15.30	52.94	297.20	0.00	2238.95
2	6	14.87	52.86	267.67	0.00	1913.73
3	4	14.33	52.76	236.69	0.01	1480.21
4	3	13.99	52.69	220.33	0.01	**1248.76**
$\beta_0 = 0.3$						
i	L	T	π_x	R	β	EAC
1	8	15.24	52.93	296.46	0.01	1400.89
2	6	14.83	52.85	267.19	0.01	975.37
3	4	14.31	52.75	236.43	0.01	381.75
4	3	13.98	52.69	220.17	0.01	**21.67**
$_0 = 0.4$						
1	8	15.18	52.92	295.72	0.01	569.73
2	6	14.79	52.84	266.72	0.01	**42.08**
3	4	14.29	52.75	236.18	0.01	-713.46
4	3	13.97	52.69	220.00	0.02	-1203.07
No feasible solution after $\beta_0 = 0.4$ and lead time 6 weeks						

Further, sensitivity analysis has also been performed with respect to α i.e. stock out risk. Results clearly indicate that as α increases (i.e. service level decreases) the target inventory level as well as the total expected cost increases as shortages occur frequently with lower level of service. The optimal solutions are also provided for different levels of service i.e. 99%, 96% and 80% in table 3, table 4 and table 5. It is clearly evident from these tables that many options for reduced lead time are available to the supplier with comparatively lower length of the review period and less cost, which implies that the reduced lead time increases the capabilities of the supplier to manage the inventory more efficiently.

Moreover, it has also been observed that when the supplier provides comparatively lower levels of service (table 4 and table 5), then reduction of lead time reduces the cost drastically even capturing a small backorder ratio. It is not reasonable for a supplier to offer high backorder discount with reduction of lead time when the level of service is comparatively low (occurrence of more shortages) which ultimately lead to infeasible solution. It is also found that supplier would like to reduce lead time on various components when back order ratio $\beta_0 > 0$.

Now, applying the algorithm 2, crashing has been carried out for lead-time for different backorder ratio when the demand during lead time is unknown and illustrated in table 6. It is observed that by reducing the lead time the total expected cost decreases where backorder ratio is positive.

Table 6: Optimal solution with service level of 98% $(\alpha = 0.02)$ for unknown distribution.

$\beta_0 = 0$						
i	L	T	π_x	R	β	EAC
1	8	12.88	52.49	464.75	0.00	5814.45
2	6	12.65	52.44	426.86	0.00	5432.43
3	4	12.38	52.39	387.35	0.00	5025.44
4	3	12.21	52.36	366.67	0.00	**4892.93**
$\beta_0 = 0.2$						
i	L	T	π_x	R	β	EAC
1	8	12.84	52.48	464.21	0.00	5364.86
2	6	12.63	52.44	426.51	0.00	4924.63
3	4	12.37	52.39	387.16	0.01	4422.52
4	3	12.21	52.36	366.54	0.01	**4212.46**
$\beta_0 = 0.5$						
i	L	T	π_x	R	β	EAC
1	8	12.80	52.47	463.41	0.01	4694.80
2	6	12.60	52.43	425.98	0.01	4166.10
3	4	12.35	52.39	386.87	0.01	3520.17
4	3	12.20	52.36	366.35	0.02	**3193.23**
$\beta_0 = 0.8$						
i	L	T	π_x	R	β	EAC
1	8	12.75	52.46	462.62	0.01	4029.82
2	6	12.57	52.43	425.46	0.02	3411.32
3	4	12.34	52.38	386.58	0.02	2620.25
4	3	12.19	52.35	366.17	0.03	**2175.75**
$\beta_0 = 1$						
i	L	T	π_x	R	β	EAC
1	8	12.72	52.46	462.09	0.02	3589.29
2	6	12.55	52.42	425.12	0.02	2910.20
3	4	12.33	52.38	386.39	0.03	2021.63
4	3	12.18	52.35	366.04	0.04	**1498.40**

Table 7: Optimal solution with service level of 99% $(\alpha = 0.01)$ for unknown distribution.

$\beta_0 = 0$						
i	L	T	π_x	R	β	EAC
1	8	12.45	52.40	550.84	0	6908.42
2	6	12.26	52.36	508.29	0	6383.87
3	4	12.03	52.32	463.75	0	5810.29
4	3	11.88	52.29	440.34	0	**5582.35**
$\beta_0 = 0.2$						
i	L	T	π_x	R	β	EAC

1	8	12.42	52.39	550.43	0.00	6602.01
2	6	12.24	52.36	508.01	0.00	6037.02
3	4	12.02	52.32	463.60	0.01	5397.37
4	3	11.87	52.29	440.25	0.01	**5115.60**
$\beta_0 = 0.5$						
i	L	T	π_x	R	β	EAC
1	8	12.39	52.39	549.81	0.01	6144.44
2	6	12.22	52.36	507.61	0.01	5518.25
3	4	12.01	52.31	463.38	0.01	4778.95
4	3	11.87	52.29	440.10	0.02	**4416.17**
$\beta_0 = 0.8$						
i	L	T	π_x	R	β	EAC
1	8	12.36	52.38	549.20	0.01	5689.30
2	6	12.20	52.35	507.21	0.02	5001.27
3	4	12.00	52.31	463.16	0.02	4161.68
4	3	11.86	52.29	439.96	0.03	**3717.58**
$\beta_0 = 1$						
i	L	T	π_x	R	β	EAC
1	8	12.34	52.38	548.80	0.02	5387.21
2	6	12.19	52.35	506.94	0.02	4657.60
3	4	11.99	52.31	463.01	0.03	3750.81
4	3	11.86	52.28	439.86	0.04	**3252.31**

Table 8: Optimal solution with service level of 96% $(\alpha = 0.04)$ for unknown distribution.

$\beta_0 = 0$						
i	L	T	π_x	R	β	EAC
1	8	13.43	52.60	405.96	0	5095.88
2	6	13.15	52.55	371.05	0	4808.10
3	4	12.82	52.48	334.68	0	4510.42
4	3	12.61	52.45	315.64	0	**4439.07**
$\beta_0 = 0.2$						
i	L	T	π_x	R	β	EAC
1	8	13.38	52.59	405.23	0.0	4429.23
2	6	13.12	52.54	370.57	0.0	4057.55
3	4	12.80	52.48	334.42	0.0	3622.94
4	3	12.60	52.44	315.48	0.0	**3440.05**
$\beta_0 = 0.5$						
i	L	T	π_x	R	β	EAC
1	8	13.31	52.58	404.15	0.01	3438.42
2	6	13.07	52.53	369.87	0.01	2938.43
3	4	12.78	52.48	334.04	0.02	2296.05

4	3	12.59	52.44	315.23	0.02	**1944.63**
$\beta_0 = 0.8$						
i	L	T	π_x	R	β	EAC
1	8	13.24	52.57	403.09	0.01	2458.37
2	6	13.03	52.53	369.17	0.02	1827.26
3	4	12.75	52.47	333.65	0.02	974.29
4	3	12.57	52.44	314.98	0.03	**452.95**
$\beta_0 = 1$						
i	L	T	π_x	R	β	EAC
1	8	13.19	52.56	402.39	0.02	1810.83
2	6	13.00	52.52	368.71	0.02	1090.81
3	4	12.74	52.47	333.40	0.03	**95.94**
4	3	12.56	52.44	314.82	0.04	-539.45
No feasible solution after lead time 4 weeks						

Table 8 represents that it is not beneficial for a supplier to crash all the components of lead time in completely backorder case because it involves extra investment of supplier where crashing of lead time is not desirable after certain point i.e. 4 weeks.

Conclusion

This study reveals the benefits associated with employment of just in philosophy i.e. reduction of lead time in periodic inventory model under the service level constraints where backorder rate is dependent on backorder discount and the length of the protection interval. The longer length of lead time implies the larger amount of shortages and the smaller proportion of customers, who can wait their order to fulfil later on, that, is why; we assume the impact of protection interval on the backorder rate. Further, the model jointly optimizes the review period, backorder discount and lead time. By controlling the lead time, supplier would be able to have higher rate of backorders and reduces the amount of shortages eventually which directly reduces the total cost of the inventory system. Two different cases of distribution i.e. Normal and unknown distribution of demand during the protection interval have been considered.

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Appendix 1:

$$H = \begin{bmatrix} \dfrac{\partial^2 EAC}{\partial T^2} & \dfrac{\partial^2 EAC}{\partial T \partial \pi_x} \\ \dfrac{\partial^2 EAC}{\partial \pi_x \partial T} & \dfrac{\partial^2 EAC}{\partial^2 \pi_x} \end{bmatrix}$$

$$EAC(T,\pi_x,L) = \frac{K + C(L_i)}{T} + \frac{\sigma\sqrt{T+L}\;\psi(A)\left[\dfrac{T\beta_0\pi_x^2}{(T+L)\pi_0} + \pi_0\left(1 - \dfrac{T\beta_0\pi_x}{(T+L)\pi_0}\right)\right]}{T}$$

$$+h\left[\frac{DT}{2} + A\sigma\sqrt{T+L} + \left(1 - \frac{T\beta_0\pi_x}{(T+L)\pi_0}\right)\sigma\sqrt{T+L}\;\Psi(A)\right]$$

$$EAC(T,\pi_x,L) = \frac{K + C(L_i)}{T} + \sigma\psi(A)\left[\frac{\beta_0\pi_x^2}{\sqrt{T+L}\;\pi_0} + \frac{\pi_0\sqrt{T+L}}{T} - \frac{\beta_0\pi_x}{\sqrt{T+L}}\right]$$

$$+h\left[\frac{DT}{2} + A\sigma\sqrt{T+L} + \left(\sqrt{T+L} - \frac{T\beta_0\pi_x}{\sqrt{T+L}\pi_0}\right)\sigma\Psi(A)\right]$$

$$\frac{\partial EAC(T,\pi_x,L)}{\partial T}=\frac{-\left[k+\sum c_j\right]}{T^2}$$

$$+\sigma\psi(A)\left[-\frac{\beta_0\pi_x^2(T+L)^{-\frac{3}{2}}}{2\pi_0}+\pi_0\left(-\frac{(T+L)^{\frac{1}{2}}}{T^2}+\frac{(T+L)^{-\frac{1}{2}}}{2T}\right)+\frac{\beta_0\pi_x(T+L)^{-\frac{3}{2}}}{2}\right]$$

$$+h\left[\frac{D}{2}+\frac{A\sigma(T+L)^{-\frac{1}{2}}}{2}+\frac{\sigma(T+L)^{-\frac{1}{2}}\psi(A)}{2}-\frac{\beta_0\pi_x\sigma\psi(A)(T+2L)(T+L)^{-\frac{3}{2}}}{2\pi_0}\right]$$

$$\frac{\partial EAC}{\partial T\partial\pi_x}=-\sigma\psi(A)\left[-\frac{\beta_0\pi_x(T+L)^{-\frac{3}{2}}}{\pi_0}+\frac{\beta_0(T+L)^{-\frac{3}{2}}}{2}-\frac{h\beta_0(T+2L)(T+L)^{-\frac{3}{2}}}{2\pi_0}\right]$$

$$\frac{\partial EAC}{\partial\pi_x}=\sigma\psi(A)\left[\frac{2\beta_0\pi_x}{\pi_0\sqrt{T+L}}-\frac{\beta_0}{\sqrt{T+L}}\right]-\frac{hT\beta_0.\sigma\psi(A)}{\pi_0\sqrt{T+L}}$$

$$\frac{\partial^2 EAC}{\partial\pi_x^2}=\frac{2\sigma\psi(A)\beta_0}{\pi_0\sqrt{T+L}}$$

$$\frac{\partial^2 EAC}{\partial T^2}=\frac{2(k+\sum c_j)}{T^3}$$

$$+\sigma\psi(A)\left[\frac{3\beta_0\pi_x^2(T+L)^{-\frac{5}{2}}}{4\pi_0}+\pi_0\left\{\frac{2(T+L)^{\frac{1}{2}}}{T^3}-\frac{(T+L)^{-\frac{1}{2}}}{T^2}-\frac{1}{4T(T+L)^{\frac{3}{2}}}\right\}-\frac{3\beta_0\pi_x(T+L)^{-\frac{5}{2}}}{4}\right]$$

$$+h\left[-\frac{A\sigma}{4}(T+L)^{-\frac{3}{2}}+\left\{-\frac{(T+L)^{-\frac{3}{2}}}{4}-\frac{\beta_0\pi_x}{2\pi_0}\left(\frac{3(T+2L)(T+L)^{-\frac{5}{2}}}{2}-(T+L)^{-\frac{3}{2}}\right)\right\}\sigma\psi(A)\right]$$

$$\frac{\partial EAC}{\partial T}=0\Rightarrow\frac{\left[k+\sum c_j\right]}{T^2}$$

$$+\sigma\psi(A)\left[\frac{\beta_0\pi_x^2(T+L)^{-\frac{3}{2}}}{2\pi_0}+\pi_0\left(\frac{(T+L)^{\frac{1}{2}}}{T^2}-\frac{(T+L)^{-\frac{1}{2}}}{2T}\right)-\frac{\beta_0\pi_x(T+L)^{-\frac{3}{2}}}{2}\right]$$

$$=h\left[\frac{D}{2}+\frac{A\sigma(T+L)^{-\frac{1}{2}}}{2}+\left(\frac{(T+L)^{-\frac{1}{2}}}{2}-\frac{\beta_0\pi_x(T+2L)(T+L)^{-\frac{3}{2}}}{2\pi_0}\right)\sigma\psi(A)\right]$$

$$\frac{\partial^2 EAC}{\partial T^2}=\frac{2(k+\sum c_j)}{T^3}+\frac{3}{4}\frac{\beta_0\pi_x^2\sigma\psi(A)(T+L)^{-\frac{5}{2}}}{\pi_0}$$

$$+\frac{2\sigma\psi(A)\pi_0(T+L)^{\frac{1}{2}}}{T^3}-\frac{\sigma\psi(A)\pi_0(T+L)^{-\frac{1}{2}}}{T^2}$$

$$-\frac{\sigma\psi(A)\pi_0}{4T(T+L)^{\frac{3}{2}}}-\frac{3\sigma\psi(A)\beta_0\pi_x(T+L)^{-\frac{5}{2}}}{4}$$

$$-\frac{hA\sigma(T+L)^{-\frac{3}{2}}}{4}-\frac{h\sigma\psi(A)(T+L)^{-\frac{3}{2}}}{4}-\frac{h\sigma\psi(A)\beta_0\pi_x}{2\pi_0}\left(\frac{3(T+2L)(T+L)^{-\frac{5}{2}}}{2}-(T+L)^{-\frac{3}{2}}\right)$$

$$>0$$

If $\beta_0\,\pi_x\,\pi_0-\beta_0\pi_x^2>0$,

this implies.

$$\therefore\frac{\partial^2 EAC}{\partial T^2}>0\quad(\because\pi_x<\pi_0)$$

i.e. $H_{11} > 0$

Further

$$\left|H_{22}\right| = \left\{ \frac{\partial^2 EAC}{\partial T^2} X \frac{\partial^2 EAC}{\partial^2 \pi_x} - \left(\frac{\partial^2 EAC}{\partial \pi_x \partial T}\right)^2 \right\}$$

$$\left\{ \begin{array}{l} \dfrac{2(k+\sum c_j)}{T^3} + \sigma\psi(A)\left[\dfrac{3\beta_0 \pi_x^2 (T+L)^{-\frac{5}{2}}}{4\pi_0} + \pi_0 \left\{ \dfrac{2(T+L)^{\frac{1}{2}}}{T^3} - \dfrac{(T+L)^{-\frac{1}{2}}}{T^2} - \dfrac{1}{4T(T+L)^{\frac{3}{2}}} \right\} - \dfrac{3\beta_0 \pi_x (T+L)^{-\frac{5}{2}}}{4} \right] \\ +h\left[-\dfrac{A\sigma}{4}(T+L)^{-\frac{3}{2}} + \left\{ -\dfrac{(T+L)^{-\frac{3}{2}}}{4} - \dfrac{\beta_0 \pi_x}{2\pi_0}\left(\dfrac{3(T+2L)(T+L)^{-\frac{5}{2}}}{2} - (T+L)^{-\frac{3}{2}} \right) \right\} \sigma\psi(A) \right] \end{array} \right\}$$

$$\times \frac{2\sigma\psi(A)\beta_0}{\pi_0 \sqrt{T+L}}$$

$$-\left(\sigma\psi(A)\left[-\frac{\beta_0 \pi_x (T+L)^{-\frac{3}{2}}}{\pi_0} + \frac{\beta_0 (T+L)^{-\frac{3}{2}}}{2} - \frac{h\beta_0 (T+2L)(T+L)^{-\frac{3}{2}}}{2\pi_0} \right] \right)^2$$

for $\pi_x < \pi_0$ Hence Proved