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Two ware-house Fuzzy Inventory model for Deteriorating Items with Ramp Type Demand and Shortages

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Abstract

In this paper a fuzzy inventory model for a single deteriorating itemwith two parameter weibulldistribution deterioration rate, ramp type demand and partial backordering at constant rate is developed. In the present market scenario due to inflation an increasing cost ofvarious factors affecting the total worth of inventory cost may increase ordering cycle length. Increase in the cost of components of inventory may not be pre-determined due to uncertainty of the market situation. Therefore, an interval based fuzzy concept has been considered to handle the situation of uncertainty. Orderingcost, holding costs in both ware-houses are assumed to be triangular fuzzy numbers. The objective of this paper is to derive an optimal replenishment policy with defuzzification of fuzzy numbers using signed distance method to minimize the average inventory cost. Further numerical examples are presented to illustrate the model. Initiallya crispmodel has been developed and then coressponding fuzzy model is derived separately and the results of twomodels are compared and analysed.

Key words : Triangular fuzzy numbers,Signed distance method, Weibull deterioration distribution,Partialbacklogging,ramp type demand, fuzzy holding cost and ordering cost.

1. Introduction

Traditionally, many models are equipped withvarious type of demand and its combination. For example stock dependent demand, constant demand. Frequent use of exponentially increasing pattern of demand depending upon time and are being followed for many decayed. It has been observed that the demand of market not always consistent with the demand pattern followed by researches in developing inventory model. In the present market scenario demand of an item may vary on the basis of its quality, availability in the market and may not follow a single pattern of demand such as demand increasing/decreasing continuously with time. Constant demand rate is only possible when life cycle of a product is at the stage of maturity. Two ware-housesinventory system has been modelled by several researcher e.g. [1-4] etc. considering variety of modelling assumptions. Some have considered demand rate is as constant through outtime period. However, as market will not consistent with a constant demand either increasing or decreasing over whole business planning horizon, so in practice onemay see variation in demand with time. The inventory model considering demand dependent on time is developed by Donaldson [5]and it is followed by many researchers. The time dependent demandis used in a two-warehouse inventory systems modelled by Goswami and Chaudhuri[6]. Also, Bhunia and Maiti [7],Banerjee and Agrawal [8] have incorporated different increasing demand pattern in their research papers developed in inventory modelling.

In the business, inventory system is not affected only by demand pattern but also utility of a product plays an important role during period of competitive market. Most Literature reveals that classical inventory models consider that utility of an item during stockremains constant but practically it is not possible in real life andutility of an item get reduced over a time period resulting deteriorationof item during the storage period. Change in the originality of an item occurs due course of time and this referred as deterioration which is a common phenomenon of an item entered in business. The deterioration of an item may occurred due to variety of reason and thus have drawn attention of researchers to incorporate this factor on priority to reduce the total inventory cost maximizing profit of business planner. Deterioration of products in inventory system has received considerable attention of researchers for manyyears. Items likeblood, alcohol, medicine,radioactive chemicals, vegitables,food andgasoline have their own life span known as Self-life and thereafter it loses its utility as time increases. There are many products in the market that are neither repaired nor replaced during marketing period.

In the period of globalization and competitive business environment, apart from the issue of deterioration in inventory system issue of limited storage is has become a practical and major problem business entity. In important and busy market,lack of large storage space may often see that forceretailers to own a small warehouse. During the seasonal period of a particular product or at the time of discount offered by distributor in festive season, retailer decides to take advantage of bulk purchase of products to increase profit in future and he buy more product over the capacity of their own house situated in the busy market place and therefore they forced for alternative arrangements to stock quantity of product purchased in bulk. To accommodate the excess quantity of products purchased, retailer hire a warehouse nearby the marketfor a shorter period on a rental basis. Since deteriorating items required special attention during stocking and therefore storage facility equipped with better facilities is required to reduce thedeterioration and hence increasing self-life period.

Due to launch of some new products e.g. fashionable items, garments, electronic items mobile etc. an uncertainty of change in any component of an inventory model may occurs and cannot be determined exactly in advance until we arrived that situation or time.For example hike in the prices, affecting the total inventory cost or hike in demand yield shortages etc. It is not easy to assess that how much? And /or when an increase/decrease in the components of an inventory model willoccurinthe future? In the business the main concerns of the management is operation decision for movement of products over time horizon that isto decide when to order? and how much to order for minimization of average inventory costs. This is more important when demand of products are uncertain and also the cost affecting inventory system are not stable. To deal uncertainty of situations, fuzzy based concept like fuzzy set theory, fuzzy numbers etc.,may be incorporated while modelling an inventory system. “Fuzzy Concept” wasfirst introduced by the Lotfi A. Zadeh [9]. Thereafter many research papers were published using fuzzy set theory considering fuzzy environment. On decision making issues in fuzzy environmentan inventory model is published by L.A.,Zadeh and R.E.Bellman [10].A fuzzy inventory model on decision making in the presence of fuzzy variables was discussed and published by R.Jain [11].On operations of fuzzy numbers, contribution of D.Dubois and H.Prade[12] can be seen in their research paper titled “some operations on fuzzy numbers”. An economic production quantity modelwith fuzzy demand and deterioration rate has been developed by SujitD.Kumar, P.K. Kund and A. Goswami[13] developed.To solvea fuzzy inventory model without shortages signed distance is applied by J.K.Syed and L.A.Aziz[14].Using triangular fuzzy number, A fuzzy inventory model accomplished without shortages is modelled by P.K.De and A.Rawat[15]. A fuzzy model with time varying demand and shortages for deteriorating items is introduced and model is developed accordingly by C.K.Jaggi, S.Pareek, A.Shrma and Nidhi[16].Fuzziness in demand is incorporated by D.Datta and Pawan Kumar [17] in the model a titled as “An optimal replenishment policy for an inventory model without shortages assuming fuzziness in demand”. Halim et. al. has incorporated fuzziness into deterioration rate of a product and developed an inventory model titled “A fuzzy inventory model for perishable items with stochastic demand,partial backlogging and fuzzy deterioration rate”.Retailer’s ordered policy and fuzziness in selling price was discussed by Goni and Maheshwari [19] in their research paper published as “Retailer’s ordered policy under the two level of delay in payments considering the demand and selling price as triangular fuzzy numbers”. Graded mean integration method is applied todefuzzify fuzzy numbers.A lot size problem in production system which unreliablewas addressed by Halim et al. [20] incorporating stochastic machine breakdown and fuzzy repair time and solved using the method of signed distance. Signed distance method was used by Singh,S.R. and Singh, C.[21] in their research paper titled as

“A fuzzy inventory model for finite rate of replenishment using signed distance method” to defuzzify fuzzy numbers.TyagiBabitaet. al. [22] has developed an inventory system with partial backordering and weibull distribution deterioration rate for two level of storage in which demand rate has been considered as constant. Singh A.S. et. al. [23] has modelledan inventory systemconsidering ramp type demand pattern and partial backordering under Weibull distribution deterioration. An optimal ordering policy for non-instantaneous deteriorating itemshas been developed by Sharma S. et. al. [24]. A fuzzy based two warehouse inventory model has been developed by Singh A.S. et al. [25] considering constant demand rate under conditionally delay in payment and model is solved using signed distance method.

OW stands for own warehouse and is abbreviated in literatures of inventory modelling concept of two warehouse is addressed with the introduction of rented warehouse in consideration with unlimited capacity that is abbreviated as RW. In general, during modelling of an inventory system researchers have considered that RWprovidesbetter stocking facilities in comparison to own ware-house and hence rate of deteriorationis reduced during storage and thus more holding cost is charged in RW. Due to higher holding cost rate,retailer forced to prefer consumption of products fromRW at earliest..

Research papers discussed above in intro and literature review, motivated to model a ware-house inventory system incorporation with general ramp-type demand rate with respect to time.Two parameter Weibull distribution deterioration rate has been considered describing practical situation of a product and effect of fuzziness is studiedwith fuzzy triangular number and model is developed accordingly. Ordering cost and holding costin both ware-houses are considered to be fuzzy triangular number andparameters are defuzzfyied using signed distance method and hence total inventory cost. This study include only single item and shortages are allowed with partial backordering at constant rate. Comparative study is done towards results of numerical example obtained from crisp and fuzzy models. Sensitive nature of model depending on selected parameters’ values are illustratedfor both models.

2.0 Definition and Preliminaries

Fuzzy inventory model need to define the following definitions:

Let set X be an universal setthenA fuzzy set A˜ on X defined as under

$$\tilde{A} = \left\{\left(x, \lambda_{\tilde{A}}(x)\right) : x \, \varepsilon \, X\right\}$$

Where $\lambda_{\tilde{A}}$: X→ $[0,1]$, is the membership function and $\lambda_{\tilde{A}}$ (x) = degree of x in A˜.

(2) The triplet (a, b, c) specify afuzzy number of triangular form; sequence of numbers is $a < b < c$. The triplet (a, b, c) is defined as

$\lambda_{\tilde{A}}$: X → $[0,1]$ presenting continuous membership function and given as under

$$\lambda_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{c-a_1} & if\ a_1 \le x \le b_1 \\ \frac{b_1-x}{c_1-b_1} & if\ c_1 \le x \le c_1 \\ 0 & otherwise \end{cases}$$

(3)Let a fuzzy set $\tilde{A}$ be defined on a set of real numbers R. Thesigned distance of fuzzy set is given by

$$d(\tilde{A},0) = \frac{1}{2}\int_0^1 [A_L(\alpha) + A_R(\alpha)] d_\alpha$$

where$A_\alpha = [A_L(\alpha) + A_R(\alpha)]\alpha \,\varepsilon\, [0,1]$ is a α-cut of a fuzzy set $\tilde{A}$.

3.0 Assumptionand Notations

The assumptions and notations used in the mathematical model are illustrated below:

3.1 Assumptions

- Ramp typedemand function is considered.
- Lead time is considered to be negligible.
- Rate of replenishment is considered infinite.
- Partial backordering is considered in case of shortages.
- Two parameter Weibull distribution deterioration rate is considered

with $\alpha, g > 0$denoting scaling factors and β,h>1denoting the shaping factors.

- Constant holding cost incorporated under condition that RW has more holding cost rate.
- During review period deteriorated units are disposed off.

Deterioration start instantaneously that is as enterdinto inventory system.

3.2 Notation

The notations are listed as under:

Demand rate per unit per unit of time

$$f(t_i) = \left\{ \begin{array}{lll} f(u) & \text{if } t_i > u & \\ & & for\ i = 1,2,3\ldots \\ f(t_i) & \text{if } t_i \le u & \end{array} \right\}$$

w	Efficacy of own warehouse
α	Deterioration rate's scaling factor of OW and $0<\alpha<1$
β	Shaping factor for deterioration in OW and $\beta>1$.
g	Deterioration rate's scaling factor of RW, $a> g$
h	Shapingfactor fordeterioration in RW and h>1.
f_d	Backordered rate

C_o	Cost of placing an order
D_c	Cost of deterioration per unit in both warehouses
H_w	Cost of holding inventory per unit per unit time in OW
H_r	Cost of holding inventory per unit per unit time in RW such that $(h_r-h_o)>0$
S_c	Cost for backlogging inventory per unit per unit of time
L_c	Lost sales (Opportunity) cost per unit
$Q_{max,i}$	Maximum order quantity at the end of cycle length for i=1,2,3
T_1	Time duration of stocked inventory in RW
T_1+T_2	Time duration of stocked inventory in Own Warehouse
T_3	Time epoch at which shortage starts
T	Total cycle length is sum of all periods i.e.$T = T_1+T_2+T_3$
$I_{ij}^{t}(t_i)$	Inventory level for the system at time t_i such that $0 \leq t_i \leq T_i$andfor casesi=1, 2, 3
R_{ti}	Positive Inventory in RW at time t_i=0for casesi= 1, 2, 3
φ^{ti}	Inventory cost per cycle for casesi= 1, 2, 3
φ^{t}	Optimal cost of inventory per cycle per unit of time Deterioration rates areas under:
t_i	Time to deterioration, $t_i>0$ Rate of deterioration in Own Warehouse

$$\theta_1(t_i)=\alpha\beta\, t_i^{(\beta-1)} \text{where } 0\leq \alpha < 1 \quad \text{and } \beta > 1$$

Rate of deterioration in Rented Warehouse

$$\theta_2(t_i)=g\, h\; t_i^{(h-1)} \text{where } 0\leq g < 1 \quad and\ h > 1,$$

{~ Sign represent the fuzziness of the parameters}

4.0 Mathematical Model

4.1 Crisp Model

During the time horizon, total cycle length is divided into three parts i.e. T_1,T_2 and T_3.Initially, a lot size inventory entered into system and a part of replenished quantity is used to fulfilbacklogged shortages, and from remaining enterspartsw units of items are stocked in the OW and excess quantity over w is kept into RW. The value of the constant parameter u with the possible values that the decision variables T_1 ,$T_2$and T_3 take, is compared and divided into three cases:

Case-1: $0\leq u \leq T_1$;Case-2: $0\leq u \leq T_2$; Case-3: $0\leq u \leq T_3$

Each case is being discussed separately.

Case-1: $0 \leq u \leq T_1$

In above case, u will attain maximum value equals to T_1.Figure-1 described situation of inventory system.The inventory level in RW deceases due to combined effect of increasing demand, deterioration during period [0 u] and due to constant demand, deterioration in the during period [u T_1].

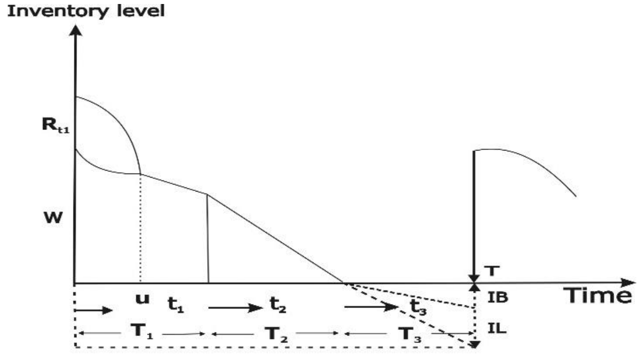


Figure 1:Inventory time graph for case-1of inventory system

Hence level of inventory is governed by the following differential equations

$$\frac{dI_{11}^{t}(t_1)}{dt_1} = -\alpha\beta t_1^{\beta-1} I_{11}^{t}(t_1) - f(t_1) ;\ 0 \leq t_1 \leq u \qquad ...(1.1)$$

$$\frac{dI_{12}^{t}(t_1)}{dt_1} = -\alpha\beta t_1^{\beta-1} I_{12}^{t}(t_1) - f(u);\ \ u \leq t_1 \leq T_1 \qquad ...(1.2)$$

With boundary conditions $I_{11}^{t}(u) = I_{12}^{t}(u)$ and $I_{12}^{t}(T_1) = 0$. The solution of 1.1 & 1.2 are resp.

$$I_{11}^{t}(t_1) = \left(f(u)\left\{(T_1 - u) + \frac{\alpha}{\beta+1}\left(T_1^{\beta+1} - u^{\beta+1}\right)\right\} - \int_{t_1}^{u} f(x) e^{\alpha x^{\beta}} dx\right) e^{-\alpha t_1^{\beta}} \qquad ...(1.3)$$

$$I_{12}^{t}(t_1) = \left(f(u)\left\{(T_1 - u) + \frac{\alpha}{\beta+1}\left(T_1^{\beta+1} - u^{\beta+1}\right)\right\}\right) e^{-\alpha t_1^{\beta}} \qquad ...(1.4)$$

In the beginning, inventory level in RW is $I_1^{t}(0) = (R_{t1} - w)$, which yields

$$R_{t1} = w + \left(f(u)\left\{(T_1 - u) + \frac{\alpha}{\beta+1}\left(T_1^{\beta+1} - u^{\beta+1}\right)\right\} - \int_{0}^{u} f(x) e^{\alpha x^{\beta}} dx\right) \qquad ...(1.5)$$

Level of inventory in Own warehousedepletes in time interval [0 T_1] due effect of deterioration only till time epoch T_1 andduring time interval [0 T_2] in OW positive stock reduces due to continuous demand at constant rate and deterioration. Thus differential equations governing the situation are illustrated as

$$\frac{dI_{13}^{t}(t_1)}{dt_1} = -ght_1^{h-1} I_{13}^{t}(t_1) ;\ 0 \leq t_1 \leq T_1 \qquad ...(1.6)$$

$$\frac{dI_{14}^{t}(t_2)}{dt_2} = -ght_1^{h-1} I_{14}^{t}(t_2) - f(t_2);\ 0 \leq t_2 \leq T_2 \qquad ...(1.7)$$

Applying boundary conditions $\mathrm{I}_3^{\mathrm{t}}(0) = \mathrm{w}$ and $\mathrm{I}_4^{\mathrm{t}}(\mathrm{T}_2) = 0$. The solution of (1.6) & (1.7) are resp.

$$\mathrm{I}_{13}^{\mathrm{t}}(\mathrm{t}_1) = \mathrm{w}e^{-g\, t_1^h} \qquad \text{...(1.8)}$$

$$\mathrm{I}_{14}^{\mathrm{t}}(\mathrm{t}_2) = \left(f(u)\left\{(\mathrm{T}_1 - \mathrm{t}_2) + \frac{g}{h+1}\left(T_1^{\beta+1} - t_2^{\beta+1}\right)\right\}\right) e^{-g\, t_2^h} \qquad \text{...(1.9)}$$

During the period of time[0 T_3] at end i.e. atT_3=0,inventory vanish in both warehouses. Continuous demand reveals shortage in the inventory system and a portion of shortage demand is backordered during next replenishmentof cycle.Present situation is described usingdifferential equation

$$\frac{d\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3)}{dt_3} = -f_d f(u); \quad 0 \le \mathrm{t}_3 \le \mathrm{T}_3 \qquad \text{...(1.10)}$$

Applying boundary condition

$\mathrm{I}_{15}^{\mathrm{t}}(0) = 0$, the solution of equation (1.10) is given as

$$\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3) = -f_d\, f(u)\mathrm{t}_3 \qquad \text{...(1.11)}$$

Inventory lost during period [0 T_3]

$$\mathrm{LS} = (1 - f_d)\ \mathrm{f}\ (\mathrm{u})\ \mathrm{T}_3$$

Total demand in period [0 T_1] a in RW is $\int_0^u f(t_1)dt_1 + \int_u^{t_1} f(u)d\,\mathrm{t}_1$

Hence quantity of inventory deteriorated during entire period is

$$\mathrm{D}_R = R_{t1} - \int_0^u f(t_1)dt_1 - \int_u^{t_1} f(u)d\,\mathrm{t}_1$$

Total demand in period$\mathrm{T}_1 + \mathrm{T}_2$ in OW is $\int_0^{T_2} f(u)d\,\mathrm{t}_2$

Hence deteriorated quantity of inventory entire period is

$$\mathrm{D}_w = w - \int_0^{T_2} f(u)d\,\mathrm{t}_2$$

Thus relevantinventory cost for entirecycle length includesbelowlisted cost factors

- Placing order costC_o
- Cost for holding Inventory in RW
- Cost for holding Inventory in OW
- Cost incurred due to Shortages
- Cost incurred due to lost sales
- Cost incurred due to deterioration in RW
- Cost incurred due to deterioration in OW

Now the present worth of total relevant average inventory cost during entirecycle length is

$$\varphi^{t1}(T_1,T_2,T_3) = \frac{1}{T}\Big[\mathrm{C}_o + H_r\left(\int_0^u \mathrm{I}_{11}^{\mathrm{t}}(\mathrm{t}_1)dt_1 + \int_0^{T_1} \mathrm{I}_{12}^{\mathrm{t}}(\mathrm{t}_1)dt_1\right) + H_w\Big(\int_0^{T_1} \mathrm{I}_{13}^{\mathrm{t}}(\mathrm{t}_2)dt_2 + \int_0^{T_2} \mathrm{I}_{14}^{\mathrm{t}}(\mathrm{t}_2)dt_2\Big) + s_c\left(\int_0^{T_3}\left(-\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3)\right)dt_3\right) + L_cLS + D_c\,(\mathrm{D}_{1R} + \mathrm{D}_{1w})\Big] \quad \ldots(1.12)$$

Case-2: $0 \leq \mathrm{u} \leq T_2$

The evolution of inventory system for above referred case is shown in **Figure-2.**During time period [0 T$_1$], level of inventory in RW reduces due course of time due todeterioration and continuous increasing demand. Therefore, inventory level describing present situation is governedby differential equation

$$\frac{d\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)}{dt_1} = -\alpha\beta t_1^{\beta-1}\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1) - \mathrm{f}(\mathrm{t}_1); \quad 0 \leq \mathrm{t}_1 \leq \mathrm{T}_1 \quad ...(2.1)$$

Applying boundary condition $\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{T}_1) = 0$. The solution of equation (2.1) is

$$\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1) = \left(\int_0^{T_1} f(x)e^{\alpha x^{\beta}}dx - \int_0^{t_1} f(x)e^{\alpha x^{\beta}}dx\right)e^{-\alpha t_1^{\beta}} \quad ...(2.2)$$

In the beginning, positive inventory in RW is $\mathrm{I}_{21}^{\mathrm{t}}(0) = (R_{t2} - w)and\mathrm{T}_1 > 0$,is as under

$$R_{t2} = \left(w + \int_0^{T_1} f(x)e^{\alpha x^{\beta}}dx)\right) > w$$

Remark-1:When$\mathrm{T}_1 = 0$, the expression obtained abovenot satisfied and level of inventory in RW vanishes. This happens when positive inventory in the beginning is less than or equal to capacity of OW. This situation describes single ware-house system and may considered as a particular case of above model.

Proceeding in similar manner in own warehouse over the periods[0 T$_1$]and [0 T$_2$] the present situation in the inventory system is governed by the following differential equations

$$\frac{d\mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_1)}{dt_1} = -ght_1^{h-1}\mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_1)\,; \quad 0 \leq \mathrm{t}_1 \leq \mathrm{T}_1 \quad ...(2.3)$$

$$\frac{d\mathrm{I}_{23}^{\mathrm{t}}(\mathrm{t}_2)}{dt_2} = -ght_2^{h-1}\mathrm{I}_{23}^{\mathrm{t}}(\mathrm{t}_2) - \mathrm{f}(\mathrm{t}_2)\,; \quad 0 \leq \mathrm{t}_2 \leq \mathrm{u} \quad ...(2.4)$$

$$\frac{d\mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)}{dt_2} = -ght_2^{h-1}\mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2) - \mathrm{f}(\mathrm{u}); \quad u \leq \mathrm{t}_2 \leq \mathrm{T}_2 \quad ...(2.5)$$

Applying boundary conditions$\mathrm{I}_{22}^{\mathrm{t}}(0) = \mathrm{w}$,$\mathrm{I}_{22}^{\mathrm{t}}(\mathrm{T}_1) = \mathrm{I}_{23}^{\mathrm{t}}(0)$ and $\mathrm{I}_{24}^{\mathrm{t}}(\mathrm{T}_2) = 0$ The solution of (2.3), (2.4) & (2.5) are respectively

$$\mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_1) = \mathrm{w}e^{-g\,t_1^h} \quad \ldots(2.6)$$

$$\mathrm{I}_{23}^{\mathrm{t}}(\mathrm{t}_2) = \left(We^{-g\,T_1^h} - \int_0^{t_2} f(x)e^{g\,x^h}dx\right)e^{-g\,t_2^h} \quad ...(2.7)$$

$$\mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2) = \left(f(u)\left\{(\mathrm{T}_2 - \mathrm{t}_2) + \frac{g}{h+1}\left(T_2^{\beta+1} - t_2^{\beta+1}\right)\right\}\right)e^{-g\,t_2^h} \quad ...(2.8)$$

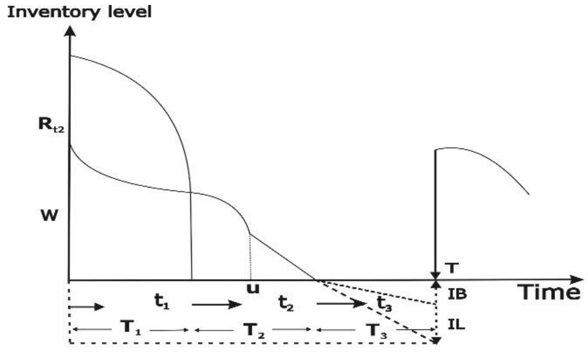


Figure- 2:Inventory time graph for case-2 of inventory system

During the time period[0 T3] and atT_3=0,level of inventory vanishes in both warehouses and shortage occurswhich is backordered at the next replenishment. This situation will be equivalent to expression given in[1.10] of case-1, *i.e.*

$$\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3) = -f_d f(u)\mathrm{t}_3 \qquad \text{...(2.9)}$$

Quantity of inventory that is lost due to not supplying inventory within periodof demand is

$$\mathrm{LS}= (1-f_d)\ \mathrm{f\,(u)\ T_3}$$

Whole demand in periodT_1 in RW is$\int_0^{T_1} f(t_1)dt_1$ which provide

The quantity of inventory deteriorated in RW is

$$\mathrm{D}_{2R} = R_{t2} - \int_0^{T_1} f(t_1)dt_1$$

The total demand during time period$\mathrm{T}_1 + \mathrm{T}_2$ at OW is $\int_0^{u} f(t_2)dt_2 - \int_u^{T_2} f(u)d\,\mathrm{t}_2$ and

Quantity of inventory deteriorated during the time period $\mathrm{T}_1 + \mathrm{T}_2$ at OW is

$$\mathrm{D}_{2w} = w - \int_0^{u} f(t_2)dt_2 - \int_u^{T_2} f(u)d\,\mathrm{t}_2$$

Now, present worth of averageinventory cost during entire cycle length is

$$\varphi^{t2}(T_1,T_2,T_3) = \frac{1}{T}\Big[\mathrm{C}_o + h_R\left(\int_0^{T_1} \mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\right) + h_w\Big(\int_0^{T_1} \mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_0^{u} \mathrm{I}_{23}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_u^{T_2} \mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Big) + s_c\left(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\right) + L_cLS + \mathrm{D}_c(\mathrm{D}_{2R} + \mathrm{D}_{2w})\Big] \qquad \text{...(2.10)}$$

Case-3: 0$\leq$ u $\leq T_3$

The inventory leveldescribing situation of case-3 is represented through Figure-3.The positive level of inventory in both warehousesbecome zero before stabilization of continuous demand. Level ofinventory at both ware-houses reduces due to deterioration and continuous growth in demandin period [0 T1] and the present situation isrepresented throughfollowing expression

$$\frac{d\mathrm{I}_{31}^{\mathrm{t}}(\mathrm{t}_1)}{dt_1} = -\alpha\beta t_1^{\beta-1}\mathrm{I}_{31}^{\mathrm{t}}(\mathrm{t}_1) - \mathrm{f}(\mathrm{t}_1);\ 0 \le \mathrm{t}_1 \le \mathrm{T}_1 \qquad \text{...(3.1)}$$

Applying boundary condition $\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{T}_1) = 0$. The solution of (3.1) is

$$\mathrm{I}_{31}^{\mathrm{t}}(\mathrm{t}_1) = \left(\int_0^{T_1} f(x)e^{\alpha x^\beta}dx - \int_0^{t_1} f(x)e^{\alpha x^\beta}dx\right)e^{-\alpha t_1^\beta}$$

In the beginning positive level of inventory in RW is $\mathrm{I}_{31}^{\mathrm{t}}(0) = (R_{t3} - W)$and $\mathrm{T}_1 > 0$,which gives

$$R_{t3} = \left(w + \int_0^{T_1} f(x)e^{\alpha x^\beta}dx)\right) > w$$

Remark 2 : When$\mathrm{T}_1 = 0$, the inequality obtained above does not hold and inventory vanishes in RW. This happens if initially, level of inventory lower to capacity of OW and situation become case of single ware-house system and may be treated as a particular case of model.

In own warehouse, positive inventory reduces over time period[0 T$_1$] due to effect of deterioration and over in period [0 T$_2$] due to deterioration and continuous growth of demand. Present situation is described through expression

$$\frac{d\mathrm{I}_{32}^{\mathrm{t}}(\mathrm{t}_1)}{dt_1} = -ght_1^{h-1}\mathrm{I}_{32}^{\mathrm{t}}(\mathrm{t}_1); \quad 0 \le \mathrm{t}_1 \le \mathrm{T}_1 \qquad \text{...(3.2)}$$

$$\frac{d\mathrm{I}_{33}^{\mathrm{t}}(\mathrm{t}_2)}{dt_2} = -ght_2^{h-1}\mathrm{I}_{33}^{\mathrm{t}}(\mathrm{t}_2) - \mathrm{f}(\mathrm{t}_2)\,; \quad 0 \le \mathrm{t}_2 \le \mathrm{T}_2 \qquad \text{...(3.3)}$$

Applying boundary conditions$\mathrm{I}_{32}^{\mathrm{t}}(0) = \mathrm{w}$, and $\mathrm{I}_{33}^{\mathrm{t}}(\mathrm{T}_2) = 0$,the solution of (3.2)& (3.3) are respectively

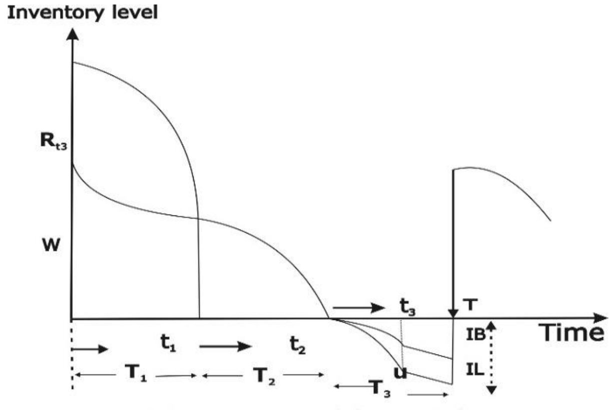


Figure -3:Inventory time graph for case-3 of Inventory system

$$\mathrm{I}_{32}^{\mathrm{t}}(\mathrm{t}_1) = \mathrm{w}e^{-g\,t_1^h} \qquad \text{...(3.4)}$$

$$\mathrm{I}_{33}^{\mathrm{t}}(\mathrm{t}_2) = \left(\int_0^{T_1} f(x)e^{gx^h}dx - \int_0^{t_1} f(x)e^{gx^h}dx\right)e^{-gt_1^h} \qquad \text{...(3.5)}$$

Further,during timeperiod[0 T$_3$]and at T$_3$=0, inventory level vanishes inboth warehouses and shortage occurs. Quantity Shortages issupplied in the beginning of next cycle length when replenished on an order. Present situations are expressed through following expression.

$$\frac{dI_{34}^{t}(t_3)}{dt_3} = -f_d f(t_3);\ \ 0 \le t_3 \le u \qquad ...(3.6)$$

$$\frac{dI_{35}^{t}(t_3)}{dt_3} = -f_d\, f(u);\quad 0 \le t_3 \le T_2 \qquad ...(3.7)$$

Applying boundary conditions$I_{34}^{t}(0) = 0$,& $I_{34}^{t}(u) = I_{35}^{t}(u)$,solutions ofexpression (3.6) & (3.7) are obtained as underrespectively

$$I_{34}^{t}(t_3) = -f_d \int_0^{t_3} f(x)dx \qquad …(3.8)$$

$$I_{35}^{t}(t_3) = -f_d\big(f(u)(t_3 - u) + \int_0^{u} f(x)dx\big) \qquad ...(3.9)$$

The quantity of inventory that is lostdue to continuous demandin period shortages is

$$\text{LS}= (1\text{-}f_d)\left(\int_0^{u} f(t_3)dt_3 + \int_0^{T_3} f(t_3)dt_3\right)$$

The total demand during time periodT_1 in RW is$\int_0^{T_1} f(t_1)dt_1$ and

Quantity of inventory deteriorated in rented warehouse is

$$D_{3R} = R_{t3} - \int_0^{T_1} f(t_1)dt_1$$

The total demand during time period$T_1 + T_2$ in OW is $\int_u^{T_2} f(t_2)\ d\,t_2$ and therefore

Quantity of inventory deteriorated during the period $T_1 + T_2$ in OW is

$$D_{3w} = w - \int_u^{T_2} f(t_2)\ d\,t_2$$

Now, the total relevant average inventory cost during entire cycle length is expressed by

$$\varphi^{t3}(T_1,T_2,T_3) = \frac{1}{T}\Big[C_o + h_R\left(\int_0^{T_1} I_{31}^{t}(t_1)dt_1\right) + h_w\left(\int_0^{T_1} I_{32}^{t}(t_2)dt_2 + \int_0^{T_2} I_{33}^{t}(t_2)dt_2\right) + s_c\left(\int_0^{u} -I_{34}^{t}(t_3)dt_3 + \int_u^{T_3} -I_{35}^{t}(t_3)dt_3\right) + L_c LS + D_c(D_{3R} + D_{3w})\Big] \qquad …(3.10)$$

Cost function$\varphi^{t}(T_1,T_2,T_3)$ can be expressedusing branch-functionsby combining equations (1.12), (2.10) & (3.10), corresponding to the three casesi.e.

$$\varphi^{t}(T_1,T_2,T_3) = \begin{cases} \varphi^{t1}(T_1,T_2,T_3) & 0 \le u \le T_1 \\ \varphi^{t2}(T_1,T_2,T_3) & 0 \le u \le T_2 \\ \varphi^{t3}(T_1,T_2,T_3) & 0 \le u \le T_3 \end{cases} \qquad ...(3.11)$$

Condition of optimality for inventory system

The minimization problem is formulated as

$$\text{Minimize:}\varphi^{t}(T_1,T_2,T_3)$$

$$\text{Subject to:}(T_1 > 0, T_2 > 0, T_3 > 0)$$

For optimal solution of model expressed by (3.11), the below given condition must hold

$$\frac{\partial\varphi^{t}(T_1,T_2,T_3))}{\partial T_1}=0;\quad \frac{\partial\varphi^{t}(T_1,T_2,T_3)}{\partial T_2}=0;\ \frac{\partial\varphi^{t}(T_1,T_2,T_3)}{\partial T_3}=0 \qquad \text{...(3.12)}$$

Solving equation (3.12) respectively for T_1 ,T_2 ,T_3 , we can obtain T_1^*, T_2^*, T_3^*, andT^* and using these values in equation (3.11) for three branches separately, total minimum inventory cost and ordering cycle length $T^* = (\, T_1^* + T_2^* + T_3^*\,)$ may be obtained.

4.2 Fuzzy Model

The model developed and expressed by equation (3.11) is a crisp model andparameters assume a specific value but it is not always be possible that parameters affecting inventory system will remain constant at any time in future too. In the present global market, the value of parameters like cost, demand may fluctuate due to the inflation and uncertainty of any reasons (like low production, natural hazards etc.) and it may fluctuate around its own value. The fluctuation at any time cannot be pre-determined until we reach the situation of that time. Therefore, the only possibility is to consider the possible range of fluctuation of values fixed. To deal with such type of uncertain situation, a fuzzy model considering vagueness of some parameter affecting the total inventory cost is developed. Values of ordering cost and holding cost in both ware-houses has been taken as fuzzy numbers which is represented by triangular fuzzy numbers. The model is solved using signed distance method to todefuzzify parameters and to minimize the total inventory costin fuzzy environment and the results of three different cases are compared with the value of crisp model. The fuzzy model for three different cases are as follows:

Case-1:

Using equation (1.12) and fuzzy parameters$\mathrm{C}_o = (\mathrm{C}_{\tilde{o}1}, \mathrm{C}_{\tilde{o}2}, \mathrm{C}_{\tilde{o}3})$,$\mathrm{H}_r = (\mathrm{H}_{\tilde{r}1}, \mathrm{H}_{\tilde{r}2}, \mathrm{H}_{\tilde{r}3})$,$\mathrm{H}_w = (\mathrm{H}_{\tilde{w}1}, \mathrm{H}_{\tilde{w}2}, \mathrm{H}_{\tilde{w}3})$we have,

$$\tilde{\varphi}^{t1}(T_1,T_2,T_3)=(\tilde{\varphi}_1^{t1}(T_1,T_2,T_3),\tilde{\varphi}_2^{t1}(T_1,T_2,T_3),\tilde{\varphi}_3^{t1}(T_1,T_2,T_3))$$

where

$$\tilde{\varphi}_1^{t1}(T_1,T_2,T_3)=\frac{1}{T}\Big[\mathrm{C}_{\tilde{o}1} + H_{\tilde{r}1}\left(\int_0^u \mathrm{I}_{11}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1 + \int_0^{T_1} \mathrm{I}_{12}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\right) + H_{\tilde{w}1}\Big(\int_0^{T_1} \mathrm{I}_{13}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_0^{T_2} \mathrm{I}_{14}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Big) + s_c\left(\int_0^{T_3}(-\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\right) + L_cLS + \mathrm{d}(\mathrm{D}_{1R} + \mathrm{D}_{1w})\Big]$$

$$\tilde{\varphi}_2^{t1}(T_1,T_2,T_3)=\frac{1}{T}\Big[f\mathrm{C}_{\tilde{o}2} + fH_{\tilde{r}2}\left(\int_0^u \mathrm{I}_{11}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1 + \int_0^{T_1} \mathrm{I}_{12}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\right) + fH_{\tilde{w}2}\Big(\int_0^{T_1} \mathrm{I}_{13}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_0^{T_2} \mathrm{I}_{14}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Big) + s_c\left(\int_0^{T_3}(-\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\right) + L_cLS + \mathrm{d}(\mathrm{D}_{1R} + \mathrm{D}_{1w})\Big]$$

$$f\tilde{\varphi}_3^{t1}(T_1,T_2,T_3)=\frac{1}{T}\Big[f\mathrm{C}_{\tilde{o}3} + fH_{\tilde{r}3}\left(\int_0^u \mathrm{I}_{11}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1 + \int_0^{T_1} \mathrm{I}_{12}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\right) + fH_{\tilde{w}3}\Big(\int_0^{T_1} \mathrm{I}_{13}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_0^{T_2} \mathrm{I}_{14}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Big) + s_c\left(\int_0^{T_3}(-\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\right) + L_cLS + \mathrm{d}(\mathrm{D}_{1R} + \mathrm{D}_{1w})\Big]$$

Thus, with above fuzzy parameters the total inventory cost in case-1 is given by the eq.

$$\tilde{\varphi}^{t1}(T_1,T_2,T_3) = \frac{1}{T}\Big[\mathrm{C}_{\tilde{o}} + H_{\tilde{r}}\Big(\int_0^{u} \mathrm{I}_{11}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1 + \int_0^{T_1} \mathrm{I}_{12}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\Big) + H_{\tilde{w}}\Big(\int_0^{T_1} \mathrm{I}_{13}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_0^{T_2} \mathrm{I}_{14}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Big) + s_c\Big(\int_0^{T_3}\big(-\mathrm{I}_{15}^{\mathrm{t}}(\mathrm{t}_3)\big)d\mathrm{t}_3\Big) + L_c LS \quad + \mathrm{d}(\mathrm{D}_{1R} + \mathrm{D}_{1w})\Big] \qquad \text{...(3.13)}$$

Similarly for other two cases, using above defined fuzzy parametersthe total inventory cost is given by following equations

Case-2 :

$$f\tilde{\varphi}^{t2}(T_1,T_2,T_3) = \frac{1}{T}\Big[f\mathrm{C}_{\tilde{o}} + fH_{\tilde{r}}\Big(\int_0^{T_1} \mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\Big) + fH_{\tilde{w}}\Big(\int_0^{T_1} \mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_u^{T_2} \mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Big) + s_c\Big(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\Big) + L_c LS + \mathrm{D}_c(\mathrm{D}_{2R} + \mathrm{D}_{2w})\Big]\text{where,} \qquad \text{...(3.14)}$$

$$f\tilde{\varphi}_1^{t2}(T_1,T_2,T_3) = \frac{1}{T}\Bigg[f\mathrm{C}_{\tilde{o}1} + fH_{\tilde{r}1}\Bigg(\int_0^{T_1} \mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\Bigg) + fH_{\tilde{w}1}\Bigg(\int_0^{T_1} \mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_u^{T_2} \mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Bigg) + s_c\Bigg(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\Bigg) + L_c LS + \mathrm{D}_c(\mathrm{D}_{2R} + \mathrm{D}_{2w})\Bigg]$$

$$f\tilde{\varphi}_2^{t2}(T_1,T_2,T_3) = \frac{1}{T}\Bigg[f\mathrm{C}_{\tilde{o}2} + fH_{\tilde{r}2}\Bigg(\int_0^{T_1} \mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\Bigg) + fH_{\tilde{w}2}\Bigg(\int_0^{T_1} \mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_u^{T_2} \mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Bigg) + s_c\Bigg(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\Bigg) + L_c LS + \mathrm{D}_c(\mathrm{D}_{2R} + \mathrm{D}_{2w})\Bigg]$$

$$f\tilde{\varphi}_3^{t2}(T_1,T_2,T_3) = \frac{1}{T}\Bigg[f\mathrm{C}_{\tilde{o}3} + fH_{\tilde{r}3}\Bigg(\int_0^{T_1} \mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\Bigg) + fH_{\tilde{w}3}\Bigg(\int_0^{T_1} \mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_u^{T_2} \mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Bigg) + s_c\Bigg(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\Bigg) + L_c LS + \mathrm{D}_c(\mathrm{D}_{2R} + \mathrm{D}_{2w})\Bigg]$$

Case-3:

$$f\tilde{\varphi}^{t3}(T_1,T_2,T_3) = \frac{1}{T}\Big[f\mathrm{C}_{\tilde{o}} + fH_{\tilde{r}}\Big(\int_0^{T_1} \mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\Big) + fH_{\tilde{w}}\Big(\int_0^{T_1} \mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2 + \int_u^{T_2} \mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\Big) + s_c\Big(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\Big) + L_c LS + \mathrm{D}_c(\mathrm{D}_{2R} + \mathrm{D}_{2w})\Big] \qquad \text{...(3.15)}$$

where

$$f\tilde{\varphi}_1^{t3}(T_1,T_2,T_3)=\frac{1}{T}\Bigg[f\mathrm{C}_{\tilde{o}1}+fH_{\tilde{r}1}\left(\int_0^{T_1}\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\right)+fH_{\tilde{w}1}\left(\int_0^{T_1}\mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2+\int_u^{T_2}\mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\right)$$
$$+\ s_c\left(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\right)+L_cLS+\mathrm{D}_c(\mathrm{D}_{2R}+\mathrm{D}_{2w})\Bigg]$$

$$f\tilde{\varphi}_2^{t3}(T_1,T_2,T_3)=\frac{1}{T}\Bigg[f\mathrm{C}_{\tilde{o}2}+fH_{\tilde{r}2}\left(\int_0^{T_1}\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\right)$$
$$+fH_{\tilde{w}2}\left(\int_0^{T_1}\mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2+\int_u^{T_2}\mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\right)+\ s_c\left(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\right)+L_cLS$$
$$+\mathrm{D}_c(\mathrm{D}_{2R}+\mathrm{D}_{2w})\Bigg]$$

$$f\tilde{\varphi}_3^{t3}(T_1,T_2,T_3)=\frac{1}{T}\Big[f\mathrm{C}_{\tilde{o}3}+fH_{\tilde{r}3}\left(\int_0^{T_1}\mathrm{I}_{21}^{\mathrm{t}}(\mathrm{t}_1)d\mathrm{t}_1\right)+fH_{\tilde{w}3}\left(\int_0^{T_1}\mathrm{I}_{22}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2+\int_u^{T_2}\mathrm{I}_{24}^{\mathrm{t}}(\mathrm{t}_2)d\mathrm{t}_2\right)+$$
$$s_c\left(\int_0^{T_3}(-\mathrm{I}_5^{\mathrm{t}}(\mathrm{t}_3))d\mathrm{t}_3\right)+L_cLS+\mathrm{D}_c(\mathrm{D}_{2R}+\mathrm{D}_{2w})\Big]$$

Combining equations (4.1), (4.2) & (4.3), the cost function $\tilde{\varphi}^t(T_1,T_2,T_3)$ of the problem results in the following three- branch function corresponding to the three casesi.e.

$$\tilde{\varphi}^t(T_1,T_2,T_3)=\begin{cases} f\tilde{\varphi}^{t1}(T_1,T_2,T_3) & 0\le \mathrm{u}\le \mathrm{T}_1\\ f\tilde{\varphi}^{t2}(T_1,T_2,T_3) & 0\le \mathrm{u}\le \mathrm{T}_2\\ f\tilde{\varphi}^{t3}(T_1,T_2,T_3) & 0\le \mathrm{u}\le \mathrm{T}_3\end{cases}\qquad \ldots(3.16)$$

Optimality condition for inventory system

The minimization problem is formulated as

$$\text{Minimize:}f\tilde{\varphi}^t(T_1,T_2,T_3)$$
$$\text{Subject to:}(T_1>0,T_2>0,T_3>0)$$

For optimal solution of model expressed by (3.16), the below given condition must hold

$$\frac{\partial f\tilde{\varphi}^t(T_1,T_2,T_3))}{\partial T_1}=0;\ \frac{\partial f\tilde{\varphi}^t(T_1,T_2,T_3)}{\partial T_2}=0;\ \frac{\partial f\tilde{\varphi}^t(T_1,T_2,T_3)}{\partial T_3}=0\qquad \ldots(3.17)$$

Solving equation (3.17) respectively for T1 ,T2 ,T3 , values of$T_1^{\tilde{}}, T_2^{\tilde{}}, T_3^{\tilde{}}$,can obtained and usingthese values of decision variables in model expression (3.16) for three branches separately, total relevant minimum inventory cost and the ordering cycle length$T^{\tilde{}}=(T_1^{\tilde{}}+T_2^{\tilde{}}+T_3^{\tilde{}})$ can be obtained.

6.0 Numerical Examples

Model is validated by considering the demand function to be exponential i.e. f(t)= Ae^{bt}, A is being demand when t=0 and b is the shapingfactor of demand function. When b receive a zero value, a constant reflected w.r.t. time and there is exponential growth in demand when b>0. The exponential function has to be solved up to first approximation. The values of parameters are realistic and chosen randomly to illustrate the model. Considering the value of parameters in an appropriate unit (displayed in Table-A) and using Mathematica-9.0software, the optimal average inventory cost has been obtained which are displayed in Table-1&Table-12 for two models separately. Sensitivity analysis is performed for both models that is for crisp model and fuzzy model on some selected parameters only.

Table-A

Parameter	A	b	U	C_o	S_c	L_c	D_o	α	β	G	h	f_d	H_r	H_w	W
Example-1	30	4.5	0.5	500	0.15	0.20	4.0	0.02	2.0	0.05	2	0.6	3.0	2.0	50
Example-2	100	3.0	1.0	1000	1.0	4.0	2.0	0.03	2.0	0.06	2	0.4	6.0	3.0	100

Example-1

Table-1 : Crisp Model:

Case	$\boldsymbol{T_1^*}$	T_2^*	$\boldsymbol{T_3^*}$	T^*	Q_{max}^* (rounded off)	$\varphi^t(T_1,T_2,T_3)$
1	0.3227	7.2985	21.3194	28.9545	34	194.88
2	0.5302	7.6269	29.5440	37.7011	35	267.05
3	0.5178	1.4836	04.9466	6.948	34	213.85

Fuzzy Model:

Case	$T_1^{\sim}$	$T_2^{\sim}$	$T_3^{\sim}$	$T^{\sim}$	$Q_{max}^{\sim}$ (rounded off)	$\tilde{\varphi}^t(T_1,T_2,T_3)$
1	0.3418	7.2933	21.9682	29.6033	24	150.43
2	0.5385	7.6237	30.0446	38.2068	36	203.58
3	0.5041	1.0800	05.9240	07.5441	32	192.05

Table-2 Variation in total inventory cost with respect to C_o

Case	C_o	T_1^*	T_2^*	T_3^*	T^*	Q_{max}^*(rounded off)	$\varphi^t(T_1,T_2,T_3)$
1	200	0.2871	7.3084	20.1123	27.7078	21	184.29
	500	0.3227	7.2985	21.3194	28.9406	34	194.88
	600	0.3342	7.3081	21.7101	29.3524	24	198.31
2	200	0.5147	7.6329	28.6260	36.7736	33	258.99
	500	0.5302	7.6269	29.5441	37.7012	35	267.05
	600	0.5315	7.6250	29.8452	38.0017	35	269.69
3	200	0.4194	1.3625	03.8423	5.6242	24	166.14
	500	0.5178	1.4836	04.9466	6.948	34	213.85
	600	0.5440	1.5177	05.2708	7.3325	36	227.85

Table-3 Variation in total inventory cost with respect to H_r

Case	H_r	T_1^*	T_2^*	T_3^*	T^*	Q_{max}^* (rounded off)	$\varphi^t(T_1,T_2,T_3)$
1	1	0.9340	7.2982	21.3720	29.6042	53	195.40
	2	0.4827	7.2981	21.3742	29.155	31	195.34
	5	0.1921	7.2998	21.1724	28.6643	53	198.10

2	1	0.9771	7.6279	29.4002	38.0052	94	265.79
	2	0.6673	7.6272	29.4997	37.7942	50	266.66
	5	0.3928	7.6266	29.5887	37.6081	22	267.41
3	1	0.9268	1.4693	5.1118	7.5079	86	208.04
	2	0.6458	1.4755	4.9014	7.0227	48	211.88
	5	0.3872	1.4886	4.9937	6.8695	22	215.88

Table-4 : Variation in total inventory cost with respect to H_w

Case	H_w	T_1^*	T_2^*	T_3^*	T^*	Q_{max}^* (rounded off)	$\varphi^t(T_1, T_2, T_3)$
1	1	0.2106	7.0602	9.3765	16.6437	9	199.40
	3	0.4858	7.5578	32.4287	40.4723	31	340.78
	4	0.5053	7.4838	38.7984	46.7875	32	348.26
2	1	0.4434	7.4828	19.1624	27.0886	27	175.95
	3	0.5731	7.6808	37.6472	45.9011	39	338.15
	4	0.5957	7.7105	44.5261	52.8323	42	398.52
3	1	0.5178	1.4836	4.9466	6.948	34	213.85
	3	0.3670	0.8445	5.6263	6.8378	20	226.82
	4	0.2443	0.7251	5.7831	6.7525	12	249.97

Table-5 : Variation in total inventory cost with respect to $\mathrm{C}_{o^{\tilde{}}}$

Case	$T_1^{\tilde{}}$	$T_2^{\tilde{}}$	$T_3^{\tilde{}}$	T^*	$f\,Q_{max}^{\tilde{}}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.3342	7.2955	21.7104	29.3401	24	148.73
2	0.5302	7.6227	29.5441	37.697	35	202.28
3	0.4861	1.0656	5.7151	7.2668	31	185.28

Table-6 : Variation in total inventory cost with respect to $\mathrm{H}_{r^{\tilde{}}}$

Case	$T_1^{\tilde{}}$	$T_2^{\tilde{}}$	$T_3^{\tilde{}}$	$T^{\tilde{}}$	$f\,Q_{max}^{\tilde{}}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.3007	7.2956	21.6888	29.2851	22	148.18
2	0.5302	7.6227	29.5441	37.697	35	202.28
3	0.4861	1.0656	5.7151	7.2668	31	185.28

Table-7 : Variation in total inventory cost with respect to $\mathrm{H}_{w^{\tilde{}}}$

Case	$T_1^{\tilde{}}$	$T_2^{\tilde{}}$	$T_3^{\tilde{}}$	$T^{\tilde{}}$	$f\,Q_{max}^{\tilde{}}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.4963	7.4653	36.6172	44.5788	32	246.84
2	0.5898	7.7022	42.3366	50.6286	41	284.45
3	0.2902	0.7596	5.7376	6.7874	14	186.00

Table-8 : Variation in total inventory cost with respect to $\mathrm{C}_{o^{\tilde{}}}$ $\mathrm{H}_{r^{\tilde{}}}$

Case	$T_1^{\tilde{}}$	$T_2^{\tilde{}}$	$T_3^{\tilde{}}$	$T^{\tilde{}}$	$Q_{max}^{\tilde{}}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.3007	7.2956	21.6828	29.2791	22	148.18
2	0.5036	7.6249	29.8555	37.984	32	202.34
3	0.4570	1.0662	05.7237	7.2469	28	185.56

Table-9 : Variation in total inventory cost with respect to f $C_{o^{\sim}}$ $H_{w^{\sim}}$

Case	$T_1^{\sim}$	$T_2^{\sim}$	$T_3^{\sim}$	$T^{\sim}$	$Q_{max}^{\sim}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.4963	7.4653	36.6172	44.5788	32	246.84
2	0.5933	7.7014	42.5612	50.8559	42	300.66
3	0.3303	0.7776	06.0910	7.1989	17	196.75

Table-10 : Variation in total inventory cost with respect to $H_{r^{\sim}}$ $H_{w^{\sim}}$

Case	$T_1^{\sim}$	$T_2^{\sim}$	$T_3^{\sim}$	$T^{\sim}$	$Q_{max}^{\sim}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.4400	7.4663	36.6172	44.5235	29	245.09
2	0.5548	7.7021	42.3466	50.6035	37	284.54
3	0.2722	0.7597	5.7401	6.772	13	186.09

Table-11 : Variation in total inventory cost with respect to $C_{o^{\sim}}$ $H_{r^{\sim}}$ $H_{w^{\sim}}$

Case	$T_1^{\sim}$	$T_2^{\sim}$	$T_3^{\sim}$	$T^{\sim}$	$Q_{max}^{\sim}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.4468	7.4653	36.6080	44.5201	29	246.78
2	0.5558	7.7014	42.5713	50.8285	38	286.02
3	0.3105	0.7764	6.0724	7.1593	16	196.86

Example-2:

Table-12 Crisp Model:

Case	T_1^*	T_2^*	T_3^*	T^*	Q_{max}^* (rounded off)	$\varphi^t(T_1, T_2, T_3)$
1	0.8341	6.8149	08.7521	16.4011	182	2356.53
2	0.2106	7.0602	09.3465	16.6173	028	2087.43
3	0.4845	1.0090	0.8298	02.3233	084	1012.20

Fuzzy Model:

Case	$T_1^{\sim}$	$T_2^{\sim}$	$T_3^{\sim}$	$T^{\sim}$	$Q_{max}^{\sim}$ (rounded off)	$\tilde{\varphi}^t(T_1, T_2, T_3)$
1	0.8418	6.6785	8.8797	16.4000	186	1785.57
2	0.8994	6.5832	8.0728	15.5554	212	1668.72
3	0.4845	1.0090	0.8298	02.3233	084	0759.15

7.0 Sensitivity Analysis

Sensitivity analysis is performed on parameter values of example-1. The following conclusionobserved from the numerically obtained results:

Crisp model

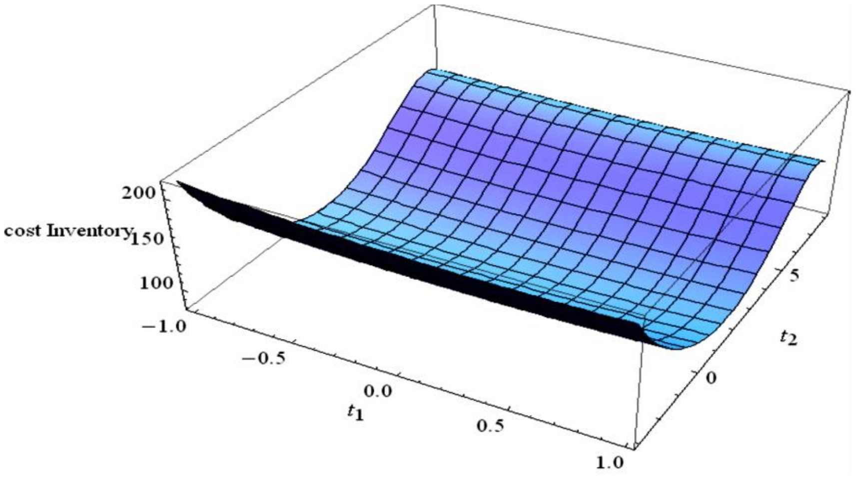
- Under given conditions and constraints, optimal solution is calculated. From Table-1 & Table-12,reveals that thepresent total relevant average inventory cost in an appropriate unit is minimum in case-1 as compared to other two cases of crisp model and in each case, the fuzzy model is less expensive than the crisp model.

- From Table-2, it is observed that in each case, the total inventory cost and cycle length are directly proportional to the ordering cost i.e. with increase in ordering cost the total inventory as well as ordering cycle length both are increases.
- From Table-3, it is observed that in each case, total inventory cost increases when holding cost in RW decreases. When holding cost increases in RW, the total inventory cost and ordering cycle length both are increases. Also, cost of average inventory is in direct proportionwith cycle length.
- From Table-4, it is observed that in each case, total inventory cost increases when holding cost in OW decreases. Cycle length is directly proportional to holding cost.

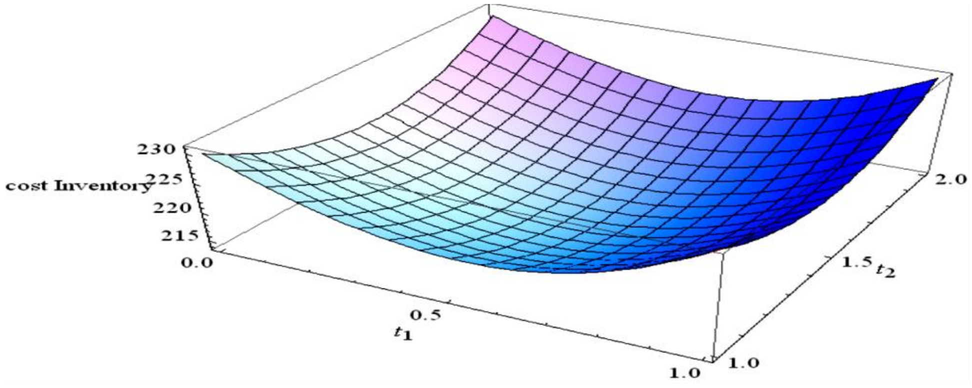
Fuzzy Model

- From Table-5, 6 & 7,it is observed that in each case, the total inventory cost decreases in fuzzy model and ordering cycle length increase while operating with single fuzzy parameter.
- From Table-8,it is observed that in each case, the total inventory cost decreases and ordering cycle length increase while operating with ordering cost and holding cost in RW as fuzzy parameter.
- From Table-9,10&11it is observed that in cases-1 & case-2, the total inventory cost increases while operating with ordering cost and holding cost in OW as fuzzy parameter and in case-3,the total inventory cost decrease but ordering cycle length increase.
- The graphs depicted in Figure-4& Figure-5 for three cases of $crisp\ model$ and $fuzzy\ model$respectively shows the convex nature of model and inventory system has a point where cost of inventory is minimum.

Case-1



Case-2



Case-3

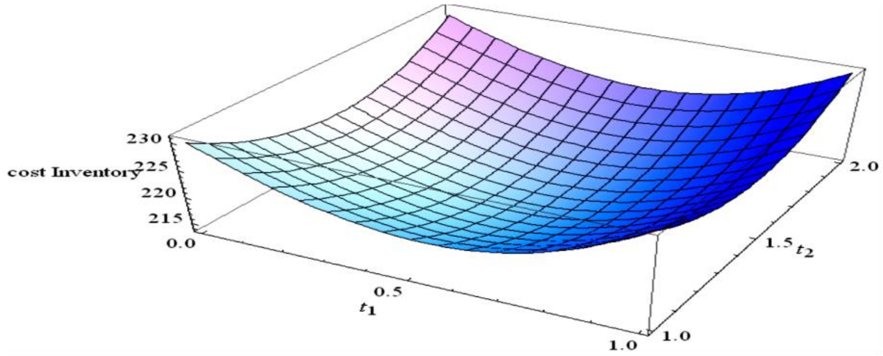
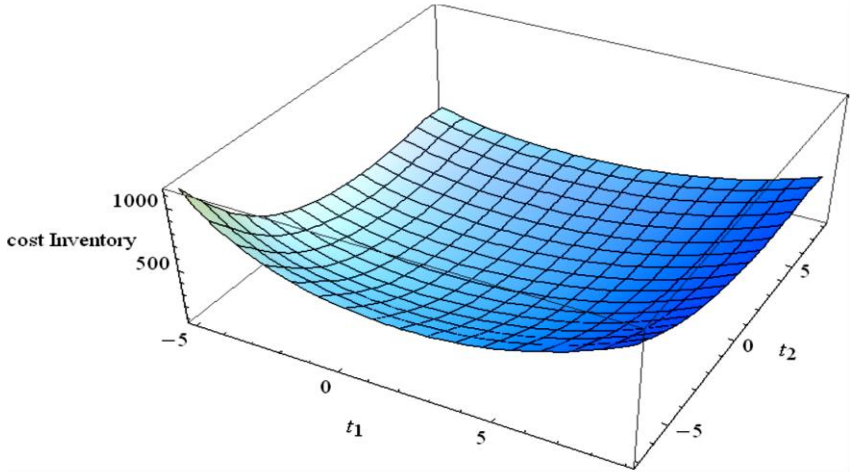
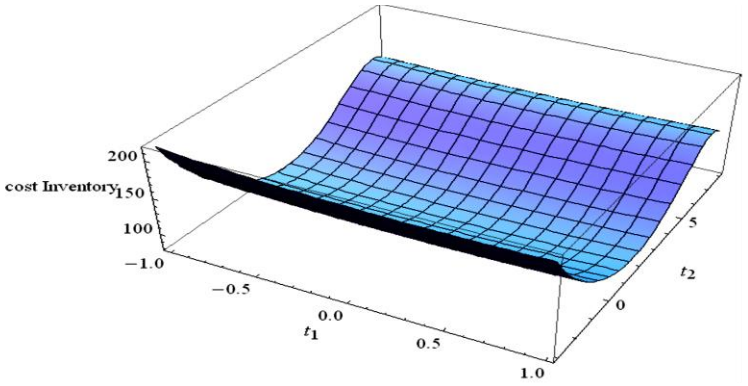


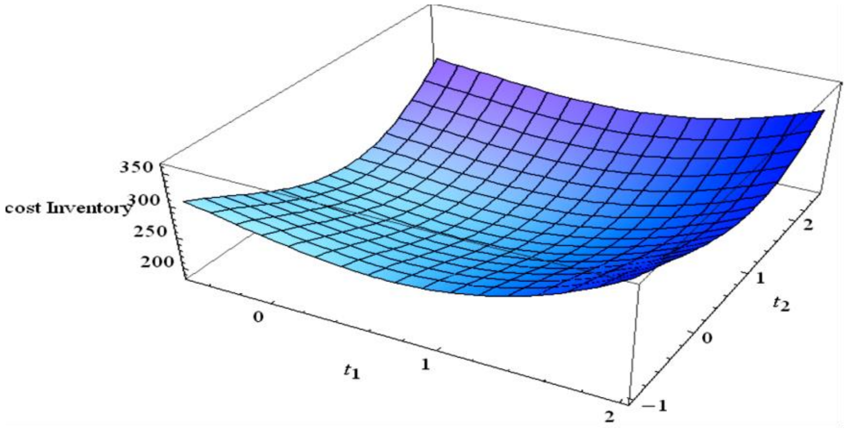
Figure-4: Graphical representation of convexity for *crisp model*



Case-1



Case-2



Case-3

Figure-5: Graphical representation of convexity for***fuzzy model***

8.0 Concluding Remarks

In this paper, a deterministic inventory model is presented to determine the optimal inventory cost for two warehouse inventory problem with ramp type demand, weibull distribution deterioration rate and partial backlogging at constant rate. The two model namely a crisp model corresponding fuzzy model is developed and numerical examples are presented to illustrate and validate theboth models. Results are obtained with the help of Mathematica 9.0 Software and compared. Sensitivity analysis is performed on some selected parameters. Fuzzy model is solved withdefuzzificationof triangular fuzzy numbers with the help of signed distance method. Further this model can be generalised by considering price dependent demand, stock dependent demand, other relevant demand and constant deterioration rate with other realistic combinations.

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